Advanced Quantum Mechanics

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Mock Exam — Institute for Theoretical Physics, University of Cologne

Support for the memory:

- $$\begin{aligned} &\sigma_0 = \hat{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ &\sigma_1 = \sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \sigma_y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{1}$$

- $$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2}$$

- $$a^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle \tag{3}$$

- $$\begin{aligned} &[A, B]_\xi = AB - (-1)^\xi BA \\ &[a_j, a_k^\dagger]_\xi = \delta_{jk}\hat{1}, \quad [a_j, a_k]_\xi = 0, \quad [a_j^\dagger, a_k^\dagger] = 0 \\ &\xi = 0 \quad \text{bosons}, \quad \xi = 1 \quad \text{fermions} \end{aligned} \tag{4}$$

- $$\partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \psi_l)} \right) + \partial_j \left(\frac{\partial \mathcal{L}}{\partial(\partial_j \psi_l)} \right) - \frac{\partial \mathcal{L}}{\partial \psi_l} = 0, \quad l = 1, \ldots, N \tag{5}$$

- $$E^2 = m^2 + \|\boldsymbol{p}\|^2 \tag{6}$$

- $$\eta = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \tag{7}$$

- $$(\partial_t^2 - \sum_j \partial_j^2 + m^2)\psi = 0 \quad \text{or equivalently} \quad (\partial^\mu \partial_\mu + m^2)\psi = 0 \tag{8}$$

- $$i\partial_t \psi = -i\alpha^j \partial_j \psi + m\beta\psi \quad \text{or equivalently} \quad (i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{9}$$

1 The reduced density operator

Consider two (distinguishable) quantum systems with Hilbert spaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$, respectively. The Hilbert space of the joint system is $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$. Suppose that the joint system is in a pure state, with the state vector

$$|\psi\rangle = \frac{1}{2}|0\rangle_1 \otimes |1\rangle_2 + \frac{\sqrt{3}}{2}|1\rangle_1 \otimes |0\rangle_2, \tag{10}$$

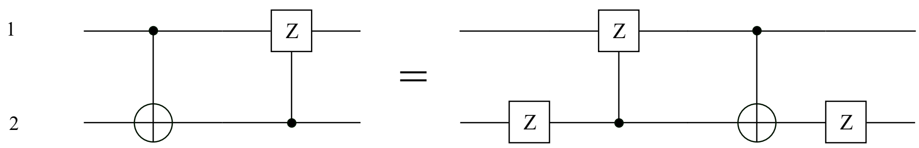
where $|0\rangle_1, |1\rangle_1$ is an orthonormal set in $\mathcal{H}^{(1)}$, and $|0\rangle_2, |1\rangle_2$ is an orthonormal set in $\mathcal{H}^{(2)}$.

a) *What is the reduced density operator of system* 1*?*
Determine whether the reduced state is mixed or pure. (Tell how you reach your conclusion.)

b) Suppose that $|0\rangle, |1\rangle$ are the eigenstates of the Pauli σ_z-operator (for both systems 1 and 2). *What is the expectation value of* $\sigma_x^{(1)} = |0\rangle_1\langle 1| + |1\rangle_1\langle 0|$*, i.e., the Pauli x operator on system* 1. The state is the same as in a).

2 Equivalence of circuits

Show the following equivalence



Hint: Recall that circuits are read from the left to the right.

3 Identical particles

Suppose that we have two identical particles along a line, with coordinates $x_1, x_2 \in \mathbb{R}$. Let $r = x_2 - x_1$ be the *relative* coordinate between the two particles. Ignoring the motion of the center of mass, the two lowest energy eigenstates of these two particles are described by the wave functions

$$\psi_a(r) = \mathcal{N}_a \cos(\beta r) e^{-\alpha r^2}, \quad \psi_b(r) = \mathcal{N}_b \sin(\beta r) e^{-\alpha r^2}, \tag{11}$$

where $\alpha > 0$, $\beta \in \mathbb{R}$, and $\mathcal{N}_a, \mathcal{N}_b$ are normalization factors. Here ψ_a corresponds to energy eigenvalue E_a, and ψ_b to energy eigenvalue E_b, with $E_a < E_b$.

a) *What is the ground state energy if the two particles are spinless bosons?*

b) *What is the ground state energy if the two particles are spinless fermions?*

c) *What is the ground state energy if the two particles are spin-half fermions? Are the two spins in a singlet-state or a triplet state, when the two particles are in the ground state?*

Justify your answers!

4 Conservation of particle-number for one-body Hamiltonians

Suppose that we have a Hamilton operator of the form

$$\hat{H} = \int \psi^\dagger(\vec{x})\vec{\nabla}^2_{\vec{x}}\psi(\vec{x})d^3x, \tag{12}$$

where $\psi(\vec{x})$ and $\psi^\dagger(\vec{x})$ are *fermionic or bosonic* field operators in the position representation. We define the total number operator as

$$\hat{N} = \int \psi^\dagger(\vec{x})\psi(\vec{x})d^3x. \tag{13}$$

Show that $[\hat{N}, \hat{H}] = 0$ in both the bosonic and fermionic case.

5 Derivation of the Klein-Gordon equation

The Klein-Gordon field has the classical Lagrange density

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right] = \frac{1}{2}\left[\partial_\mu\phi g^{\mu\nu}\partial_\nu\phi - m^2\phi^2\right] \tag{14}$$

Use the Euler-Lagrange equations to derive the equation of motion. Note that we here use the Einstein summation convention. *When quantizing this theory, what is the particle statistics of the quantized exica-tions?*

6 Dirac equation

The Dirac equation can be written in the form

$$i\frac{\partial}{\partial t}\Psi = -i\sum_{l=1}^{3}\alpha^l\partial_l\Psi + m\beta\Psi, \tag{15}$$

where Ψ is a spinor, and where we have put $\hbar = 1$ and $c = 1$.

Make an ansatz in the form of plane waves in (15)

$$\Psi_{\vec{p}}(t,\vec{r}) = we^{-i[E_{\vec{p}}t - \vec{p}\cdot\vec{r}]}, \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}. \tag{16}$$

By inserting $\Psi_{\vec{p}}$ into (15), and using the representation where

$$\alpha^k = \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{bmatrix}, \tag{17}$$

the Dirac equation (15) reduces to an eigenvalue problem for the vectors w, i.e, $Mw = E_{\vec{p}}w$.

a) *Determine the matrix M.* (It is enough to express this matrix in terms of the Pauli matrices σ_k and the identity matrix $\mathbb{I}$.)

b) *Determine an orthonormal set of eigenvectors of M for particles at rest, i.e., plane-wave solutions with momentum $\vec{p} = 0$. Identify the positive and negative energy solutions.*