Advanced Quantum Mechanics

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Exercise sheet 10 **Due: Sunday June 18 at 24:00**

1 Coherent states of oscillators and fields

As discussed briefly in section 3.3.2 of the lecture notes, coherent field states somehow resembles classical fields. For quantum harmonic oscillators, the evolution of coherent states shadows that of the corresponding classical oscillator. Since the free EM field can be regarded as an infinite collection of harmonic oscillators, it is maybe not surprising that there is a similar correspondence. Here we shall explore this. Apart from the physical relevance, coherent states are also very useful, and pop up in all kinds of places, so it is in any case good to know about them.

For a single mode, the family of coherent states can be written

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

for any $\alpha \in \mathbb{C}$.

a) *What is the overlap $\langle\alpha|\beta\rangle$? Express $|\langle\alpha|\beta\rangle|^2$ in terms of $|\alpha-\beta|^2$. What does this say about the approximate orthogonality of $|\alpha\rangle$ and $|\beta\rangle$ as $|\alpha-\beta|^2$ grows?* **(2 points)**

b) Recall that the quantum vector potential is

$$\boldsymbol{A}(\boldsymbol{x},t) = \sqrt{\frac{\hbar}{\epsilon_0 L^3}} \sum_{\boldsymbol{k},\lambda} \frac{\boldsymbol{e}_\lambda(\boldsymbol{k})}{\sqrt{2\omega_k}} (a_{\boldsymbol{k},\lambda} e^{-i\omega_k t + i\boldsymbol{k}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{k},\lambda} e^{+i\omega_k t - i\boldsymbol{k}\cdot\boldsymbol{x}}).$$

Now consider a quantum EM field such that each $\boldsymbol{k},\lambda$-mode is in a coherent state $|\alpha_{\boldsymbol{k},\lambda}\rangle$. In other words, the total state is[1]

$$|\chi\rangle = \Pi_{\boldsymbol{k}',\lambda'} |\alpha_{\boldsymbol{k}',\lambda'}\rangle. \quad (1)$$

Determine the expectation value $\langle\chi|\boldsymbol{A}(\boldsymbol{x},t)|\chi\rangle$. This expectation value is nothing but a classical vector potential of a free field. *Confirm this by showing that $\langle\chi|\boldsymbol{A}(\boldsymbol{x},t)|\chi\rangle$ satisfies the Maxwell equation of the free-field vector potential, $(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\langle\chi|\boldsymbol{A}(\boldsymbol{x})|\chi\rangle = 0$, and that it satisfies the Coulomb gauge condition $\nabla \cdot \langle\chi|\boldsymbol{A}(\boldsymbol{x})|\chi\rangle = 0$.*

Hint: Recall that the polarization vectors $\boldsymbol{e}_\lambda(\boldsymbol{k})$ are orthogonal to the direction of propagation $\boldsymbol{k}$, i.e., $\boldsymbol{e}_\lambda(\boldsymbol{k}) \cdot \boldsymbol{k} = 0$. Moreover, recall that $a|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle\alpha|a^\dagger = \alpha^*\langle\alpha|$. **(4 points)**

Remark: This shows that the expectation value of the quantum vector potential with respect to the coherent field state follows the evolution of the classical counterpart. Does this mean that field coherent states precisely correspond to classical fields? No, not quite, and that has to do with the intrinsic uncertainties in the quantum case. For the harmonic oscillator, the coherent states are minimal uncertainty states. The field coherent states have similar properties, as we shall see in problem c).

[1] One may note that we here operate in the Heisenberg picture, where $\boldsymbol{A}$ evolves and the state $|\chi\rangle$ is fixed.

c) Analogous to what we did in problem 2 on sheet 9, let us investigate the fluctuations of the force $\boldsymbol{F} = \int \rho(\boldsymbol{x})\hat{\boldsymbol{E}}(\boldsymbol{x})d^3x$, on a charge distribution $\rho(\boldsymbol{x})$, for the electric field operator (and as in sheet 9, we only consider $t = 0$)

$$\boldsymbol{E}(\boldsymbol{x}) = i\sqrt{\frac{\hbar}{\epsilon_0 L^3}}\sum_{\boldsymbol{k},\lambda}\sqrt{\frac{\omega_k}{2}}\boldsymbol{e}_\lambda(\boldsymbol{k})\left(a_{\boldsymbol{k},\lambda} - a^\dagger_{-\boldsymbol{k},\lambda}\right)e^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

with respect to the coherent field state $|\chi\rangle$ as in (1).

Show that

$$\Delta F := \langle\chi|\,\|\boldsymbol{F}\|^2\,|\chi\rangle - \|\langle\chi|\boldsymbol{F}|\chi\rangle\|^2 = \frac{\hbar}{2\epsilon_0 L^3}\sum_{\boldsymbol{k},\lambda}\omega_k|\tilde{\rho}(\boldsymbol{k})|^2.$$

with $\tilde{\rho}(\boldsymbol{k}) = \int \rho(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}d^3x$.

Hint: Recall that $\boldsymbol{e}_\lambda(-\boldsymbol{k}) = \boldsymbol{e}_\lambda(\boldsymbol{k})$ and $\omega_{-k} = \omega_k$. There is a massive cancellation of terms between $\langle\chi|\,\|\boldsymbol{F}\|^2\,|\chi\rangle$ and $\|\langle\chi|\boldsymbol{F}|\chi\rangle\|^2$.

(5 points)

Remark: We can conclude that the fluctuations will be non-vanishing in general. Note also that these fluctuations are independent of the choice of coherent field state, and that they are the same as those that we obtained for the vacuum state $|vac\rangle$ in problem 2 on sheet 9. One might note that the vacuum state actually is a coherent state.

Something to think about: The field coherent state $|\chi\rangle$ is determined by the collection of complex numbers $\alpha_{\boldsymbol{k},\lambda}$. Suppose that we would use the very same numbers to determine a classical electric field. What would the variance of the force be?[2]

2 Light-matter interactions: Jaynes-Cummings via the rotating wave approximation

The Jaynes-Cummings model is a particularly simple model of an atom interacting with the electromagnetic field. This model is particularly suitable for describing a situation where an atom is inside a cavity, and where the frequency of a mode corresponding to a standing wave in the cavity matches (precisely or closely) the energy level difference between two energy levels in the atom. More precisely, we let $|0\rangle$ denote the ground state of the atom and $|1\rangle$ the selected excited state. With $\hbar\omega_{atom} = E_1 - E_0$ being the energy gap, we describe the Hamiltonian of these two levels as $H_{atom} = \hbar\frac{\omega_{atom}}{2}\sigma_z$, where $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. The Hamiltonian of the select mode of the cavity is modeled as $H_{cavity} = \hbar\omega_{cavity}a^\dagger a$. Finally, we assume that the two-level system and the field mode interact via the Hamiltonian

$$H_I = \hbar g(a^\dagger + a)\sigma_x,$$

where g is some constant and $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$.[3] Although already a quite simple model, it can nevertheless be quite messy to analyze this Hamiltonian. Here we are going to use the rotating-wave approximation in order to find an even simpler model when $\omega_{cavity} \approx \omega_{atom}$, i.e., at (near) resonance.

[2] If one consider an alternative where $\alpha_{\boldsymbol{k},\lambda}$ are random variables, then one can get a nonzero variance of the force also in the classical case. This corresponds to models of noisy fields, e.g., due to thermal fluctuations.

[3] One may certainly wonder where this interaction Hamiltonian comes from. Here we will simply take it for granted, but one may compare with the final result in section 3.4 in the lecture notes. With g_{ijk} being all zero except for $k =$ cavity and $(i,j) = (0,1)$ or $(i,j) = (1,0)$, and $g_{0,1,\text{cavity}} = g_{1,0,\text{cavity}} = \hbar g$, we get $\hbar g(a_k^\dagger + a_k)(b_1^\dagger b_0 + b_0^\dagger b_1)$. One can view $b_1^\dagger b_0 + b_0^\dagger b_1$ as the second quantized version of $\sigma = |0\rangle\langle 1| + |1\rangle\langle 0|$.

a) The first step is to transform from the Schrödinger picture to the interaction picture. Let us for a moment consider a more general Hamiltonian $H = H_0 + H_1$, where H_0 is time-independent (and where H_1 may or may not be time-dependent). Suppose that $|\psi(t)\rangle$ is a solution to the Schrödinger equation $i\hbar \frac{d}{dt}|\psi(t)\rangle = \big(H_0 + H_1\big)|\psi(t)\rangle$, and define the new states $|\psi(t)\rangle_{\text{int}} = e^{itH_0/\hbar}|\psi(t)\rangle$. *Show that*

$$i\hbar\frac{d}{dt}|\psi(t)\rangle_{\text{int}} = H_{1,\text{int}}(t)|\psi(t)\rangle_{\text{int}}, \quad H_{1,\text{int}}(t) = e^{itH_0/\hbar}H_1 e^{-itH_0/\hbar}.$$

(3 points)

b) *Apply the interaction picture to $H_0 = H_{cavity} + H_{atom}$ and $H_1 = H_{\text{I}}$ and show that*

$$\begin{aligned} H_{\text{I,int}}(t) =& \hbar g e^{-it(\omega_{cavity}+\omega_{atom})} a\sigma_- + \hbar g e^{-it(\omega_{cavity}-\omega_{atom})} a\sigma_+ \\ &+ \hbar g e^{it(\omega_{cavity}-\omega_{atom})} a^\dagger\sigma_- + \hbar g e^{it(\omega_{cavity}+\omega_{atom})} a^\dagger\sigma_+, \end{aligned}$$

where $\sigma_+ = |1\rangle\langle 0|$ and $\sigma_- = |0\rangle\langle 1|$.

Hint: recall that $e^{A+B} = e^A e^B$ if $[A, B] = 0$. It could be useful to evaluate $e^{it\omega_{cavity}a^\dagger a} a e^{-it\omega_{cavity}a^\dagger a}$ and $e^{it\omega_{cavity}a^\dagger a} a^\dagger e^{-it\omega_{cavity}a^\dagger a}$, as well as $e^{it\frac{\omega_{atom}}{2}\sigma_z}\sigma_x e^{-it\frac{\omega_{atom}}{2}\sigma_z}$.

(3 points)

c) By assuming that $\omega_{cavity} \approx \omega_{atom}$ it follows that the terms $e^{\pm it(\omega_{cavity}-\omega_{atom})}$ revolves very slowly compared to the more fast oscillating terms $e^{\pm it(\omega_{cavity}+\omega_{atom})}$. The rotating wave approximation is to ignore the fast oscillating terms.[4] *Apply these approximations, transform back to the Schrödinger picture, and show that*

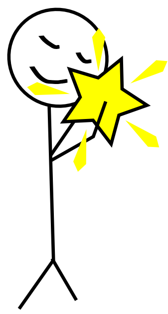
$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H'|\psi(t)\rangle, \qquad H' = \hbar\omega_{cavity}a^\dagger a + \hbar\frac{\omega_{atom}}{2}\sigma_z + \hbar g a\sigma_+ + \hbar g a^\dagger\sigma_-.$$

This Hamiltonian is the Jaynes-Cummings model. **(3 points)**

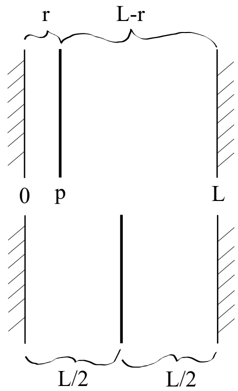
Remark: To understand the Jaynes-Cummings Hamiltonian H', one might note that the term $a\sigma_+$ excites the two-level system from the ground state $|0\rangle$ to the excited state $|1\rangle$, and simultaneously removes one quantum from the harmonic oscillator. The term $a^\dagger\sigma_-$ conversely de-excites the two-level system and adds a quantum to the oscillator. Hence, a single quantum of energy is moved back and forth between the oscillator and the two-level system.

[4] One might certainly ask why this is a reasonable approximation. To get a hint, one can rewrite the Schrödinger equation $i\hbar\frac{d}{dt}|\psi(t)\rangle_{\text{int}} = H_{\text{I,int}}(t)|\psi(t)\rangle_{\text{int}}$ as the integral equation $|\psi(t)\rangle_{\text{int}} = |\psi(0)\rangle_{\text{int}} + \int_0^t H_{\text{I,int}}(s)|\psi(s)\rangle_{\text{int}}ds$. If we for the sake of simplicity assume $\omega_{cavity} = \omega_{atom}$, then we have the two terms $\int_0^t a\sigma_+|\psi(s)\rangle_{\text{int}}ds$ and $\int_0^t a^\dagger\sigma_-|\psi(s)\rangle_{\text{int}}ds$ (those that we keep) and we have the two terms $\int_0^t e^{-2is\omega_{cavity}}a\sigma_-|\psi(s)\rangle_{\text{int}}ds$ and $\int_0^t e^{2is\omega_{cavity}}a^\dagger\sigma_+|\psi(s)\rangle_{\text{int}}ds$. If we now *assume* that $|\psi(s)\rangle_{\text{int}}$ varies very slowly compared to the oscillations of $e^{\pm 2is\omega_{cavity}}$, then the resulting integral would be very small, since the oscillations would average to something close to zero. Hence, it could potentially make sense to cancel these two oscillatory terms. However, to turn this argument into an actual proof, one would need to establish bounds on how fast $|\psi(s)\rangle_{\text{int}}$ actually changes with s.

3 Gold star exercise: The Casimir force



In this exercise we explore a particular aspect of light-matter interactions, where the quantum nature of the system really makes itself noticed, namely the Casimir force. This is closely related to the ground state energy of the electromagnetic field inside cavities. As you have seen, the electromagnetic field can be described as a collection of harmonic oscillators. Usually, the harmonic oscillator has the Hamiltonian $\hbar\omega_k(a_k^\dagger a_k + \frac{1}{2}\hat{1})$. However, when we construct the Hamiltonian of the field, we have to sum over an infinite number of modes. In order to avoid the infinity $\sum_k \frac{1}{2}\hbar\omega_k\hat{1}$, we simply throw these terms away. We do this with the rationale that for each finite set of modes, this only corresponds to a subtraction of a constant from all energy levels, which has no observable consequences. (See the discussions in section 3.1 of the lecture notes.)
In view of this one may wonder whether the 'vacuum energy' $\sum_k \frac{1}{2}\hbar\omega_k\hat{1}$ only is a mathematical oddity without any physical meaning, and thus can be ignored completely. The existence of the Casimir force suggests that the situation maybe is not quite as simple as that. The point is that the gauge invariance of physics under the subtraction of a constant energy is only valid if the subtracted energy indeed truly is a constant. If the ground state energy can *change*, then observable effects may emerge. Here we shall investigate this in a simplified one-dimensional model of the electromagnetic field in a cavity, where we indeed do change the ground state energies of the field by modifying the cavity.



Consider one-dimensional 'cavity' of length L, where we insert a partition p that divides the cavity into two parts. We thus get two cavities, one of length r and one of length $L-r$. Now consider the cavity of length r. Standing waves in that cavity can be formed at frequencies $\omega_k = \frac{k\pi}{r}$ for $k = 1, \ldots$. On these modes we have the annihilation operators $a_{k,\lambda}$ and creation operators $a_{k,\lambda}^\dagger$ (where λ is the polarization), and we get the Hamiltonian $H = \sum_{k,\lambda} \hbar\omega_k(a_{k,\lambda}^\dagger a_{k,\lambda} + \frac{1}{2}\hat{1})$. For the vacuum state $|\mathrm{vac}\rangle$, the expectation value of the energy is
$E(r) = \langle \mathrm{vac}|H|\mathrm{vac}\rangle = \hbar\sum_k \omega_k = \hbar\sum_k \frac{\pi k}{r}$. For the combined cavities r and $L-r$, the vacuum energy thus is $E(r) + E(L-r)$, which we know is infinite. Recall that absolute energies often do not have much meaning, but that energy differences do. So, let us compare with the case where we insert the wall in the middle of the cavity.[5] The difference in energy is $\Delta E(r) = E(r) + E(L-r) - 2E(L/2)$. However, we do not really gain much, since we in this case have an expression of the form $\infty + \infty - 2\infty$. In order to handle this, we will in the following regularize these infinities by introducing a cutoff-function $g(\omega) = e^{-\xi\omega}$, where the regularized energy function is[6]

$$E(r,\xi) = \hbar\sum_{k=1}^{\infty}\omega_k g(\omega_k) = \hbar\sum_{k=1}^{\infty}\frac{\pi k}{r}e^{-\xi\frac{\pi k}{r}}.$$

The function $E(r,\xi)$ is finite for all $\xi > 0$, and $E(r,0) = E(r)$. The role of the cutoff-function is to gradually cut off the high frequencies. As ξ goes to zero, the higher up in frequency we move the cutoff.

a) A good thing about this particular cutoff function is that we can find a closed expression for

[5] There is nothing special with putting the wall in the middle. Any fixed reference position would do.
[6] Do you think that we are cheating with this regularization business? Well, welcome to the world of quantum field theory!

the regularized energy. *Show that*

$$E(r,\xi) = \frac{\hbar\pi}{r}\frac{e^{-\xi\frac{\pi}{r}}}{(1-e^{-\xi\frac{\pi}{r}})^2}.$$

(0 points)

b) We are interested in what happens for small values of ξ (i.e., when the cutoff only significantly affects very high frequencies).

Show that

$$E(r,\xi) = \alpha\frac{r}{\xi^2} + \beta\frac{1}{r} + O(\xi)$$

and determine the constants α *and* β. **(0 points)**

c) We define the regularized energy difference

$$\Delta E(r,\xi) = E(r,\xi) + E(L-r,\xi) - 2E(L/2,\xi).$$

We now wish to determine this energy difference as a function of $r > 0$ in the limit where we take $\xi \to 0$ (thus take the cutoff to infinity), and where we make the larger cavity infinitely large $L \to \infty$ (so that we get the effective force between p and wall 0). *Show that*

$$\lim_{L\to\infty}\lim_{\xi\to 0}\Delta E(r,\xi) = -\frac{\gamma}{r},$$

and determine the constant $\gamma > 0$. *What is thus the effective force between the plate* p *and the wall* 0*, as a function of of the distance* r*? Is the force attractive or repulsive between the wall and the plate?*

(0 points)

Remark: With a more accurate model, where one treats the walls and the plate as extended surfaces, one finds that the force is proportional to r^{-4}. The existence of the Casimir effect has been confirmed by experiments.