Advanced Quantum Mechanics

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Exercise sheet 11 **Due: Sunday June 25 at 24:00**

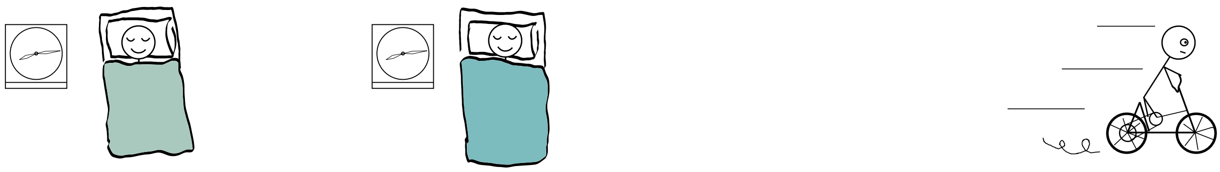
1 A quick reminder about special relativity

This exercise is to remind you about the basics of special relativity. I should also say sorry for temporarily re-introducing c as the standard speed of light, in spite of the first being said in the lecture notes is that we put $\hbar = c = 1$.

The time and space coordinates for one and the same event do generally depend on the relative speed of the observers, and the relation is determined by the Lorentz transformation. Suppose that observer S assigns coordinates (ct, x, y, z) to a particular event. If observer S' moves with speed v in the x-direction relative to observer S, then S' assigns the coordinates (ct', x', y', z') given by

$$t' = \left(t - \frac{vx}{c^2}\right)\gamma(v), \quad x' = (x - vt)\gamma(v), \quad y' = y, \quad z' = z, \quad \gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

a) Students Alice and Bob sleep happily in their beds[1] on Tuesday morning shortly before 8:15. We assume that their beds are not moving relative to each other (and are not subject to any significant accelerations). Their beds stand 5 meters apart in their respective rooms in their WG. In their frame of reference, their alarm clocks go off with a time-difference of 5ns. Student Charlie is very eager to reach the lecture on advanced quantum mechanics in time for 8:15, and is already on his bike, traveling at speed v relative to Alice and Bob. Moreover, it so happens that Charlie travels parallel to the line joining Alice and Bob. From the perspective of Charlie, the two alarm-clocks go off at the same time. *What is the speed v?*[2] *What is the distance between Alice and Bob from Charlies perspective?* **(2 points)**



b) The energy momentum-relation for a particle with rest mass m and momentum p is

$$E_{\vec{p}} = \sqrt{c^2p^2 + m^2c^4}, \quad p = \|\vec{p}\|. \tag{1}$$

You may also recall that the apparent mass $\tilde{m}(v)$ and the momentum p of a particle from the point of view of an observer moving with the speed v relative to the particle are

$$\tilde{m}(v) = \gamma(v)m, \quad p = \tilde{m}(v)v = \gamma(v)mv. \tag{2}$$

Since the speed is limited by c, does this mean that the magnitude p of the momentum also has to be bounded? What happens with the total energy $E_{\vec{p}}$ as v approaches c? Use (1) and (2) to show that $E_{\vec{p}} = \gamma(v)mc^2$. **(3 points)**

[1] Alice and Bob have sold their startup Cologne Quantum Supercomputing. Unfortunately, they did not sell it until after the recent credit-crunch, so they basically only got the worth of a curry-wurst each for it. However, they are anyway rather happy, since they now can sleep in peace again.

[2] Well, maybe I am stretching things a bit, but Charlie really is in a hurry.

2 Equations of motions of fields via Lagrangians

As you may recall, Newton's mechanics can be rephrased as Lagrangian mechanics, where each system has a corresponding Lagrange function, and we obtain the equations of motion via the Euler-Lagrange equations. You will most likely have covered the case of a finite number of degrees of freedom. An example is a single particle with mass m that moves in $\mathbb{R}^3$ with coordinates q^1, q^2, q^3 in a potential $U(\boldsymbol{q})$. The corresponding Lagrange-function is $L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{k=1}^{3} \frac{m}{2} \dot{q}_k^2 - U(\boldsymbol{q})$, and the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0, \quad k = 1,2,3, \tag{3}$$

yield the equations of motion $m\ddot{\boldsymbol{q}} = -\nabla U(\boldsymbol{q})$. However, there are systems that have an infinite number of degrees of freedom, like vibrating strings, or more general fields. In these cases, the Lagrange function L is replaced by a Lagrangian *density* $\mathcal{L}$ which is a function of a collection of fields $\psi_l(\boldsymbol{x}, t)$ for $l = 1, \ldots, N$. The density $\mathcal{L}$ can also depend on all the time-derivatives $\partial_t \psi_l$ (or $\partial_0 \psi_l$, which is a short-hand notations for $\frac{d\psi_l}{dt}$), as well as all the spatial derivatives $\partial_j \psi_l$ (short-hand notation for $\frac{d\psi_l}{dx^j}$). The corresponding Euler-Lagrange equations are [3]

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \psi_l)} \right) + \partial_j \left(\frac{\partial \mathcal{L}}{\partial(\partial_j \psi_l)} \right) - \frac{\partial \mathcal{L}}{\partial \psi_l} = 0, \quad l = 1, \ldots, N, \tag{5}$$

where we apply the Einstein summation convention, so that $\partial_j \left(\frac{\partial \mathcal{L}}{\partial(\partial_j \psi_l)} \right)$ means $\sum_j \partial_j \left(\frac{\partial \mathcal{L}}{\partial(\partial_j \psi_l)} \right)$. In case you wish to understand how (5) comes about, we derive it in the gold-star exercise, using variational calculus (Hamilton's principle). However, in the following, we shall simply apply (5) on a few examples. Note that in this exercise we regard the fields merely as real (or complex) valued functions; no funky operator-valued fields in this exercise. We are moreover back to $\hbar = c = 1$.

a) Let us start with a field in $1D$ with only one component, and we let the Lagrangian density be[4]

$$\mathcal{L} = \frac{\sigma}{2}(\partial_t \psi)^2 - \frac{\kappa}{2}(\partial_x \psi)^2, \tag{6}$$

where $\sigma > 0$ and $\kappa > 0$ are constants. *Show that the E-L equation* (5) *yields the wave-equation*

$$\sigma \partial_t^2 \psi - \kappa \partial_x^2 \psi = 0.$$

(2 points)

b) Consider now a field with two components ψ_1 and ψ_2, but still in one dimension. This field has the Lagrandgian density

$$\mathcal{L} = -\psi_1 \partial_t \psi_2 + \psi_2 \partial_t \psi_1 - \frac{1}{2m}(\partial_x \psi_1)^2 - \frac{1}{2m}(\partial_x \psi_2)^2. \tag{7}$$

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[3] I don't know if it helps, but if we avoid all the short-hand notations for the derivatives, then (5) would be written

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \frac{d\psi_l}{dt}} \right) + \frac{d}{dx^j}\left(\frac{\partial \mathcal{L}}{\partial \frac{d\psi_l}{dx^j}} \right) - \frac{\partial \mathcal{L}}{d\psi_l} = 0, \quad l = 1, \ldots, N. \tag{4}$$

Equation (5) (or (4)) is actually much less scary than it may look at first sight, and after one has got over the somewhat crazy-looking notation, it is actually rather efficient. To see the analogy between (5) and (3), one may note that the term $\frac{\partial \mathcal{L}}{d\psi_l}$ corresponds to $\frac{\partial L}{\partial q_k}$, while both $\partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \psi_l)} \right)$ and $\partial_j \left(\frac{\partial \mathcal{L}}{\partial(\partial_j \psi_l)} \right)$ in some sense correspond to $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}$. One may also note that the position $\boldsymbol{x}$ is no longer a dynamical variable, but is rather a *parameter*, on equal footing with time t.

[4] Without the short-hand notations, (6) would read

$$\mathcal{L} = \frac{\sigma}{2}\left(\frac{d\psi}{dt}\right)^2 - \frac{\kappa}{2}\left(\frac{d\psi}{dx}\right)^2.$$

Use the E-L equations in order to determine the equations of motion for ψ_1, ψ_2. **(2 points)**

c) Now, consider a *complex* field ψ in one dimension, with the Lagrangian density

$$\mathcal{L} = \frac{i}{2}\psi^*\partial_t\psi - \frac{i}{2}\psi\partial_t\psi^* - \frac{1}{2m}(\partial_x\psi^*)(\partial_x\psi). \tag{8}$$

In the following, we regard ψ and ψ^* (as well as their derivatives) as independent variables of $\mathcal{L}$. In other words, we have a two-component field with the two components ψ and ψ^*. *Use the E-L equations in order to determine the equations of motion for ψ, ψ^*. Do you recognize the equation of motion for ψ?*

(3 points)

d) *With ψ_1 and ψ_2 being real-valued functions, show that with $\psi = \psi_1 + i\psi_2$, equation (8) can be rewritten as (7). Also, show that the equations of motion for ψ_1, ψ_2 that you found in b) yield the equations of motion for ψ, ψ^* that you found in c).*

(2 points)

Remark: b) and c) illustrates the fact that a complex field ψ either can be described via its real and imaginary components ψ_1 and ψ_2, or as ψ together with its complex conjugate ψ^*. Which version to use is largely a matter of taste and convenience.

e) We let $\psi(\boldsymbol{x}, t)$ be a real-valued field with the Lagrangian density of the real Klein-Gordon field

$$\mathcal{L} = \frac{1}{2}(\partial^\alpha\psi)(\partial_\alpha\psi) - \frac{1}{2}m^2\psi^2 \equiv \frac{1}{2}\eta^{\alpha\beta}(\partial_\beta\psi)(\partial_\alpha\psi) - \frac{1}{2}m^2\psi^2,$$

where we now use the Einstein summation convention with $\alpha, \beta = 0, 1, 2, 3$, and where η is the metric tensor introduced in the lecture, and $m > 0$ is just a number.

Hint: It might be convenient to introduce $x^0 = t$ and rewrite (5) as

$$\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi_l)}\right) - \frac{\partial\mathcal{L}}{d\psi_l} = 0, \quad l = 1, \ldots, N, \tag{9}$$

where the Einstein convention now runs over $\nu = 0, 1, 2, 3$. *Use the E-L equation in to obtain the equation of motion.*

(3 points)

f) let $\psi(\boldsymbol{x}, t)$ be a complex field with Lagrangian density of the complex Klein-Gordon field

$$\mathcal{L} = \frac{1}{2}(\partial^\alpha\psi^*)(\partial_\alpha\psi) - \frac{1}{2}m^2\psi^*\psi \equiv \frac{1}{2}\eta^{\alpha\beta}(\partial_\beta\psi^*)(\partial_\alpha\psi) - \frac{1}{2}m^2\psi^*\psi.$$

Use the E-L equation in to obtain the equations of motion.

(3 points)

3 Gold star exercise: Euler-Lagrange equations for fields

In this exercise we are going to derive the Euler-Lagrange equations (5) from the Lagrange density $\mathcal{L}$. Let us first recall that the Euler-Lagrange equations (3) for finite degrees of freedom are obtained by finding the stationary point of the action functional $S = \int L dt$. The continuum case is treated in an analogous manner, but where the action integral now spans over both space and time

$$S = \int \int \mathcal{L} d^3 x dt. \tag{10}$$

One may note that we equivalently could write this as $S = \int L dt$, where $L = \int \mathcal{L} d^3 x$. In other words, we obtain the Lagrangian L by integrating the Lagrangian density $\mathcal{L}$ over space (which thus explains the name 'Lagrangian density').

Let $(\psi_l)_{l=1}^N$ be a stationary point to S, and let $(\xi_l)_{l=1}^N$ be some functions that all vanish at the boundary of the integration region (wherever that boundary is). We define

$$\psi_l(\alpha) = \psi_l + \alpha \xi_l,$$

with $\alpha \in \mathbb{R}$. With this choice of $\psi_l(\alpha)$ in (10), the action S becomes a function of α. Since $(\psi_l)_{l=1}^N$ is assumed to be stationary, it thus follows that $\frac{d}{d\alpha} S(\alpha) = 0$. *Show that this condition implies the Euler-Lagrange equations* (5).

Hint: Use the chain-rule on $\mathcal{L}$, where you regard $\mathcal{L}$ as being a function of all ψ_l, of all $\partial_t \psi_l$, and all $\partial_j \psi_l$. Next, use partial integration on some of the terms in $\int \int \frac{d}{d\alpha} \mathcal{L} d^3 x dt$. It would be more compact to use the notation in (9) rather than the version in (5).

(0 points)