Advanced Quantum Mechanics

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Exercise sheet 12 **Due: Sunday July 2 at 24:00**

1 The non-relativistic limit of the Klein-Gordon equation

In the lecture we introduce the Klein-Gordon equation for free particles, where we put $c = \hbar = 1$. However, in this exercise we temporarily re-dress the KG equation to its full glory as

$$\frac{1}{c^2}\partial_t^2\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0. \tag{1}$$

The reason for why we re-introduce c and $\hbar$ is that we want to take the non-relativistic limit, and for this, it turns out to be easier to interpret things if one keeps $\hbar$ and c.[1]

In the lecture we first treat ψ as a wave-function, and later as an operator acting on the Hilbert space of a quantum field. In this exercise we will follow the wave-equation scenario, while in the next exercise we work with the field version.

As mentioed above, we wish to analyze the behavior of the KG equation in the non-relativistic limit. To this end, it is useful to rewrite the KG equation, which is second order in time, into two coupled equations that are first order in time[2].

a) For any function[3] $\psi(t,\vec{r})$ we define two new functions $\phi(t,\vec{r})$ and $\chi(t,\vec{r})$ by

$$\begin{aligned}\phi(t,\vec{r}) =& \frac{1}{2}\psi(t,\vec{r}) + \frac{i\hbar}{2mc^2}\partial_t\psi(t,\vec{r}),\\ \chi(t,\vec{r}) =& \frac{1}{2}\psi(t,\vec{r}) - \frac{i\hbar}{2mc^2}\partial_t\psi(t,\vec{r}).\end{aligned} \tag{2}$$

Show that if ψ satisfies the KG equation, then ϕ and χ satisfy

$$i\hbar\partial_t\phi = -\frac{\hbar^2}{2m}\nabla^2(\phi+\chi) + mc^2\phi, \qquad i\hbar\partial_t\chi = \frac{\hbar^2}{2m}\nabla^2(\phi+\chi) - mc^2\chi. \tag{3}$$

Hint: At some point it can be useful to invert (2) and express ψ and $\partial_t\psi$ in terms of ϕ and χ. **(3 points)**

Remark: To show that the coupled equations (3) are equivalent to the KG equation (1), we should strictly speaking also show that if ϕ and χ satisfy (3), then $\psi = \phi + \chi$ satisfy (1). However, we skip this (although nothing would prevent you from showing it anyway :-)

b) *Make an ansatz of the form*

$$\begin{bmatrix}\phi(t,\vec{r})\\ \chi(t,\vec{r})\end{bmatrix} = e^{-\frac{i}{\hbar}(Et-\vec{p}\cdot\vec{r})}\begin{bmatrix}a\\ b\end{bmatrix}, \quad E\in\mathbb{R}, \quad a,b\in\mathbb{C},$$

in (3) and show that this leads to an eigenvalue problem of the form $M\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right] = E\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right]$ for a 2×2 matrix M. Determine M, and find its eigenvalues, and argue why we should expect to get these eigenvalues.

(4 points)

[1] At least I think its easier to interpret.

[2] Transformations between equations with higher order derivatives, and coupled equations with lower order derivatives is a common trick that can be rather useful.

[3] Well, for any sufficiently smooth function.

c) *Determine the eigenvectors of M, and combine this with b) to write down the corresponding solutions to (3) as*

$$\Psi_{\pm}(t,\vec{r}) = \begin{bmatrix} \phi(t,\vec{r}) \\ \chi(t,\vec{r}) \end{bmatrix}_{\pm} = \mathcal{N} e^{-\frac{i}{\hbar}(\pm E_{\vec{p}}t - \vec{p}\cdot\vec{r})} \begin{bmatrix} mc^2 \pm E_{\vec{p}} \\ mc^2 - \pm E_{\vec{p}} \end{bmatrix},$$

where $E_{\vec{p}} = \sqrt{c^2p^2 + m^2c^4}$, $\quad p = \|\vec{p}\|$, *with m being the rest mass of the particle.* The quantity $\mathcal{N}$ is a normalization factor that we do not bother to determine.

(2 points)

d) We can conclude from b) and c) that the free Klein-Gordon equation has two types of plane-wave solutions. One class where the energy is positive, and one where the energy is negative. Often these are somewhat vaguely associated to particles and anti-particles. (With a field theoretic treatment, as in exercise 2, we do not have to be vague any more.) A rather relevant question is how these solutions behave in the low energy limit, i.e., when speeds are not relativistic. In particular one can note that the two components of the vector $\begin{bmatrix} mc^2 \pm E_{\vec{p}} \\ mc^2 - \pm E_{\vec{p}} \end{bmatrix}$ determines the relative weight between ϕ and χ in the solutions $\Psi_{\pm}$.

- *What are the weights of the two components* ϕ *and* χ *for the positive and negative plane-waves* Ψ_+ *and* Ψ_- *in the case when the momentum is zero?*
- *Expand* $E_{\vec{p}}$ *up to the first order in* $\frac{p^2}{m^2c^2}$. *You will get two energy terms. Interpret these two terms.*
- *What happens to* $\begin{bmatrix} mc^2 \pm E_{\vec{p}} \\ mc^2 - \pm E_{\vec{p}} \end{bmatrix}$ *for small* $\frac{p^2}{m^2c^2}$? *What does that mean for the relative weight of* ϕ *and* χ *in the solutions* $\Psi_{\pm}$?

(4 points)

e) *Argue that the evolution of positive energy states in the non-relativistic regime (i.e. for small* $\frac{p^2}{m^2c^2}$*) is approximately governed by a Schrödinger equation.* In other words, show that we in the non-relativistic limit regain what we are used to in non-relativistic quantum mechanics.

Hint: Consider the results in d) for this regime. Which terms in (3) are going to be large, and which are going to be small? Be bold and only consider the equation for the dominant term, and put the small things to zero in that equation. Note that this problem to its very nature is rather hand-wavy, so we do not expect any particularly rigorous arguments. **(3 points)**

2 Two conserved quantities of the KG equation that actually are the same

Here we turn to the field-version of the Klein-Gordon equation, where the wave-functions are re-interpreted as the field operators

$$\begin{aligned}\phi(\boldsymbol{x}) &= \int \sqrt{\frac{1}{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-ip_\mu x^\mu} + b^\dagger_{\boldsymbol{p}} e^{ip_\mu x^\mu} \right) \frac{d^3\boldsymbol{p}}{(2\pi)^{3/2}} \\ &= \int \sqrt{\frac{1}{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-i(E_{\boldsymbol{p}}t - \boldsymbol{p}\cdot\boldsymbol{x})} + b^\dagger_{\boldsymbol{p}} e^{i(E_{\boldsymbol{p}}t - \boldsymbol{p}\cdot\boldsymbol{x})} \right) \frac{d^3\boldsymbol{p}}{(2\pi)^{3/2}}\end{aligned}$$

where we, like in the lecture, have put $\hbar = 1$ and $c = 1$. Moreover, $a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}$ and $b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{p}}$ are bosonic annihilation and creations operators for particles and anti-particles. In terms of these annihilation and creation operators, one can define the charge operator

$$Q = \sum_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} - \sum_{\boldsymbol{p}} b^\dagger_{\boldsymbol{p}} b_{\boldsymbol{p}}.$$

In the following we will show that this operator is almost, up to one of those annoying infinities, the same as[4]

$$R = i \int \left(\phi^\dagger(\partial_t \phi) - (\partial_t \phi^\dagger)\phi \right) d^3r.$$

Show that

$$R = \int a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} d^3p - \int b_{\boldsymbol{p}} b^\dagger_{\boldsymbol{p}} d^3p.$$

Note the 'wrong' ordering in $b_{\boldsymbol{p}} b^\dagger_{\boldsymbol{p}}$.

Hint: For the calculations you can assume that $\frac{1}{(2\pi)^3} \int e^{i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{x}} d^3x = \delta_{\boldsymbol{p},\boldsymbol{p}'}$.

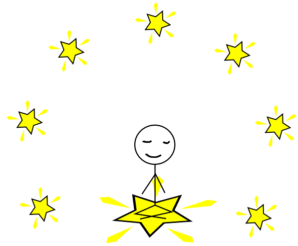
(4 points)

Remark: By the canonical commutation relations it follows that $b_{\boldsymbol{p}} b^\dagger_{\boldsymbol{p}} = b^\dagger_{\boldsymbol{p}} b_{\boldsymbol{p}} + \hat{1}$. In other words, R is equal to Q, up to a term that diverges to infinity. [5]

[4] In the gold star exercise we suse Noether's theorem in order to show how this type of conserved quantity comes about.

[5] Sometimes people use the notion of 'normal ordering' as a tool to deal with these infinities, often denoted $:Q:$, where Q is some product of annihilation and creation operators. The normal ordering $:Q:$ then means that we move all creation operators to the left of all annihilation operators. For example, $:a_1 a_2^\dagger a_1^\dagger := a_2^\dagger a_1^\dagger a_1$. In this notation, we would thus have $:R:= Q$. Although it gives a compact way of dealing with this type of infinities, it does not really explain anything.

3 Gold star exercise: R via Noether's theorem



In exercise 2, the quantity R kind of just fell from the sky. Here we shall see how the non-quantized counterpart (for complex-valued fields rather the operator valued ones)

$$R = i\int (\psi^*(\partial_t\psi) - (\partial_t\psi^*)\psi)\, d^3r.$$

comes about via Noether's theorem. By Noether's theorem we can associate conserved quantities to symmetries of the Lagrangian of the system.

You will most likely have encountered this in classical mechanics for systems with finite degrees of freedom, but it also works for fields. We will not derive Noether's theorem for fields here, but simply just apply it in order to see that R actually emerges as a consequence of a very basic symmetry of the KG field. One version of Noether's theorem goes as follows: Suppose that a Lagrangian density $\mathcal{L}$ is invariant under the mapping $\psi_l \mapsto \psi_l^{(s)}$, where s is some real parameter, and $\psi_l^{(0)} = \psi_l$, then

$$F = \int \frac{\partial\mathcal{L}}{\partial(\partial_t\psi_l)}\frac{d\psi_l^{(s)}}{ds}\bigg|_{s=0} d^3x$$

is a conserved quantity (where we sum over l).

As you may recall from sheet 11, the Lagrangian density of the KG field is

$$\mathcal{L} = \frac{1}{2}(\partial^\alpha\psi^*)(\partial_\alpha\psi) - \frac{1}{2}m^2\psi^*\psi \equiv \frac{1}{2}(\partial_t\psi^*)(\partial_t\psi) - \frac{1}{2}(\nabla\psi^*)\cdot(\nabla\psi) - \frac{1}{2}m^2\psi^*\psi, \tag{4}$$

where we regard ψ and ψ^* as two independent fields.

a) *Show that (4) is invariant under the mapping*

$$(\psi,\psi^*) \mapsto (\psi^{(s)},\psi^{*(s)}) = (e^{is}\psi, e^{-is}\psi^*).$$

(0 points)

b) *What is the corresponding conserved quantity?*

Hint: Since the ψ and ψ^* are complex-valued, rather than operator-valued, we have $\psi^*(\partial_t\psi) = (\partial_t\psi)\psi^*$, i.e,, the ordering does not matter. Moreover, multiplying a conserved quantity with a non-zero constant yields an equivalent conserved quantity.

(0 points)