Advanced Quantum Mechanics

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Exercise sheet 13 Solution

1 Equation of continuity for the Dirac equation

The Dirac equation can be written as

$$i\frac{\partial}{\partial t}\psi = -i\alpha^j\partial_j\psi + m\beta\psi,$$

where we apply the summation convention, and where $\alpha^1, \alpha^2, \alpha^3$, and β are the 4×4 matrices defined in the lecture. One should moreover keep in mind that ψ is a column vector with four components (a spinor). We define the density $\rho = \psi^\dagger\psi$ and the current $\boldsymbol{j} = \psi^\dagger\boldsymbol{\alpha}\psi$, which means $\boldsymbol{j} = (j^1, j^2, j^3) = (\psi^\dagger\alpha^1\psi, \psi^\dagger\alpha^2\psi, \psi^\dagger\alpha^3\psi)$. *Show that*

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\boldsymbol{j} = 0.$$

(4 points)

2 Plane wave solutions to the Dirac equation

For the Dirac equation for a free particle, we do in the lecture make the plane wave ansatz $\psi = \boldsymbol{u}_p e^{-ip_\mu x^\mu} = \boldsymbol{u}_p e^{-iEt+i\boldsymbol{p}\cdot\boldsymbol{x}}$, where the vectors $\boldsymbol{u}_p$ turns out (see section 4.3.1 in the lecture notes) to be solutions to the equation

$$\begin{bmatrix}(E-m)I & -\boldsymbol{\sigma}\cdot\boldsymbol{p}\\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & (-E-m)I\end{bmatrix}\boldsymbol{u}_p = 0. \tag{1}$$

We also observe that when $\boldsymbol{p} = 0$, then positive energy solutions correspond to the two first components in $\boldsymbol{u}_p$, while the two last correspond to negative energy solutions. We shall see in this exercise that which components that correspond to positive or negative solutions depend on $\boldsymbol{p}$. Here we let $\boldsymbol{p}$ be directed in the z-direction, i.e., $\boldsymbol{p} = (0,0,p)$. Consider the following four vectors

$$\boldsymbol{u}_+^{(+)}(p) = \begin{bmatrix}1\\0\\\frac{p}{E_p+m}\\0\end{bmatrix},\quad \boldsymbol{u}_+^{(-)}(p) = \begin{bmatrix}0\\1\\0\\-\frac{p}{E_p+m}\end{bmatrix},\quad \boldsymbol{u}_-^{(+)}(p) = \begin{bmatrix}-\frac{p}{E_p+m}\\0\\1\\0\end{bmatrix},\quad \boldsymbol{u}_-^{(-)}(p) = \begin{bmatrix}0\\\frac{p}{E_p+m}\\0\\1\end{bmatrix}, \tag{2}$$

where $E_p = \sqrt{m^2+p^2}$.

Show that these four vectors are solutions to (1), and determine the corresponding values of E. Which of these vectors correspond to positive or negative energies?

(4 points)

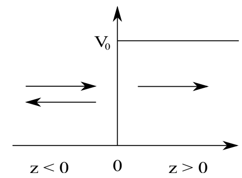
Remark: You will note that some of these vectors correspond to the same energy. However, it turns out that these instead have different spin-properties. One can define an angular momentum operator as $\boldsymbol{\Sigma} = \left[\begin{smallmatrix}\boldsymbol{\sigma} & \\ & \boldsymbol{\sigma}\end{smallmatrix}\right]$, as well as its projection onto the direction of the momentum (often referred to as the helicity) as $\boldsymbol{\Sigma}\cdot\boldsymbol{p} = \left[\begin{smallmatrix}\boldsymbol{\sigma}\cdot\boldsymbol{p} & \\ & \boldsymbol{\sigma}\cdot\boldsymbol{p}\end{smallmatrix}\right]$. It turns out that $\boldsymbol{\Sigma}\cdot(0,0,p)\boldsymbol{u}_s^{(\xi)}(p) = \xi p\boldsymbol{u}_s^{(\xi)}(p)$. Hence, each vector in (1) not only correspond to a well defined energy, but also a well-defined helicity.

3 The Klein step

When a non-relativistic quantum particle (i.e., governed by the Schrödinger equation) impinges on a potential step it gets reflected if the kinetic energy is lower than the height of the potential, although the wave function has an exponentially decaying tail that extends into the classically forbidden region. Oddly enough, for the Dirac equation, we will find solutions with a non-zero current inside the step , even thought the energy is too low. The Dirac equation in a step potential is sometimes referred to as the Klein step.[1]

We consider a potential $V(z) = 0$ for $z < 0$ and $V(z) = V_0$ for $z > 0$. For a particle of rest mass m we wish to find stationary solutions with a well defined total energy E. As an ansatz we divide the wave-function into $\psi_{z<0}$ for the region $z < 0$ and $\psi_{z>0}$ for $z > 0$, and for $\boldsymbol{p} = (0, 0, p)$, $\boldsymbol{q} = (0, 0, q)$ and $p > 0$ we let

$$\begin{aligned} \psi_{z<0}(z) =& e^{ipz} u_+^{(+)}(p) + A e^{-ipz} u_+^{(+)}(-p), \\ \psi_{z>0}(z) =& B e^{iqz} u_+^{(+)}(q), \end{aligned} \tag{3}$$



where $u_+^{(+)}$ is the vector in (2), and where A and B are yet undetermined coefficients.

a)
- What is the direction of motion, as well as the sign of the energy of $e^{ipz} u_+^{(+)}(p)$ and $e^{-ipz} u_+^{(+)}(-p)$?
- Argue that it must be the case that $q^2 = \frac{1}{c^2}(E - V_0)^2 - m^2 c^2$.
- For which values of the energy E is q a real number, and for which values is it imaginary?
- Relate the cases of real and imaginary q to plane-waves and decaying solutions in the region $z > 0$. (In the imaginary case, we need to choose a suitable sign.) For a fixed total energy E, argue that the solution becomes a plane-wave in the region $z > 0$ for all sufficiently large V_0.

(5 points)

b) Our guess in (3) contains the undetermined coefficients A and B. We determine these by demanding that the solution is continuous at $z = 0$, i.e., we assume that $\psi_{z<0}(0) = \psi_{z>0}(0)$. Show that

$$A = \frac{1-\eta}{1+\eta}, \quad B = \frac{2}{1+\eta}, \quad \text{where} \quad \eta = \frac{q}{p} \frac{E+m}{E - V_0 + m}.$$

(2 points)

c) From exercise 1 we can recall the definition of the current density $\boldsymbol{j} = (j_1, j_2, j_3)$. Here we are interested in the z-component of the current of the incoming, reflected, and transmitted components of the above solution. Show that for both imaginary and real q it is the case that these currents balance each other, such that $j_{3,\text{in}} + j_{3,\text{refl}} = j_{3,\text{trans}}$, where one should keep in mind to keep the signs of the j_3s. **(3 points)**

d) From c) we see that the currents balance as one would expect. However, things are still odd. Show that if $V_0 > E + m$ and $q > 0$, then $|j_{3,\text{refl}}| > |j_{3,\text{in}}|$, and $j_{3,\text{trans}} < 0$. In other words, show that the reflected current is larger than the incoming current, and the transmitted current is negative. **(2 points)**

Remark: In the literature, these strange currents are (somewhat vaguely) attributed to creation of particles and anti-particles. However, for this type of reasoning we would need to employ a field-theoretic interpretation of the Dirac equation.

[1] The Klein step is very closely related to the Klein paradox, where one instead of a step has a square potential barrier. However, that is more messy to analyze, so that was why we only do the step here.