Advanced Quantum Mechanics

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Exercise sheet 2 **Due: Sunday April 16 at 24:00**

1 Angular momentum, singlet and triplet states

Angular momentum is in classical mechanics described as a vector $\vec{j} = (j_x, j_y, j_z)$ pointing in the direction of the rotation axis. In quantum mechanics we do instead describe angular momentum via a vector of operators $\vec{J} = (J_x, J_y, J_z)$. For spin-half systems these angular momentum operators take the form

$$J_x = \frac{1}{2}\hbar\sigma_x, \quad J_y = \frac{1}{2}\hbar\sigma_y, \quad J_z = \frac{1}{2}\hbar\sigma_z.$$

The Pauli operators σ_x, σ_y, σ_z are often expressed in terms of their matrix representations (the Pauli matrices) $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ with respect to the eigenbasis basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ of σ_z, where the two orthonormal eigenvectors $|\uparrow\rangle$ and $|\downarrow\rangle$ correspond to the eigenvalues $+1$ and -1, respectively.

a) Show that the Pauli operators can be written as

$$\sigma_x = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|, \quad \sigma_y = -i|\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow|, \quad \sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|.$$

(2 points)

b) Suppose now that we have *two* spin-half systems with Pauli-operators σ_{1x}, σ_{1y}, σ_{1z} and σ_{2x}, σ_{2y}, σ_{2z}, respectively. Show that

$$\sigma_{1x} \otimes \sigma_{2x} + \sigma_{1y} \otimes \sigma_{2y} = 2|\uparrow\downarrow\rangle\langle\downarrow\uparrow| + 2|\downarrow\uparrow\rangle\langle\uparrow\downarrow|.$$

(3 points)

Remark: We have here used a compact notation where, e.g., $|\uparrow\uparrow\rangle$ is the same as $|\uparrow\rangle_1|\uparrow\rangle_2$, which also can be written $|\uparrow\rangle_1 \otimes |\uparrow\rangle_2$. Consequently, $|\uparrow\downarrow\rangle\langle\downarrow\uparrow|$ can be rewritten as $|\uparrow\rangle_1\langle\downarrow| \otimes |\downarrow\rangle_2\langle\uparrow|$.

c) For two angular momentum operators $\vec{J}_1 = (J_{1x}, J_{1y}, J_{1z})$ and $\vec{J}_2 = (J_{2x}, J_{2y}, J_{2z})$, the total angular momentum operator is given by $\vec{J} = \vec{J}_1 \otimes \hat{1}_2 + \hat{1}_1 \otimes \vec{J}_2$. Show that for two spin-half systems it is the case that

$$\vec{J}^2 = \frac{3}{2}\hbar^2\hat{1}_1 \otimes \hat{1}_2 + \frac{1}{2}\hbar^2\sigma_{1z} \otimes \sigma_{2z} + \hbar^2(|\uparrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow|).$$

(3 points)

d) Show that the "singlet state"

$$|\psi_{0,0}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle),$$

and the "triplet states"

$$|\psi_{1,1}\rangle = |\uparrow\uparrow\rangle, \quad |\psi_{1,0}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad |\psi_{1,-1}\rangle = |\downarrow\downarrow\rangle,$$

are simultaneous eigenvectors of the two operators $\vec{J}^2$ and $J_z = J_{1z} \otimes \hat{1}_2 + \hat{1}_1 \otimes J_{2z}$, and determine the corresponding eigenvalues. **(4 points)**

2 Density operators and the Bloch representation

Density operators generalize the notion of quantum states, such that instead of being represented by normalized vectors in a Hilbert space, they correspond to Hermitian positive semidefinite operators with trace 1 on the Hilbert space. The familiar case when the state can be represented by a vector $|\psi\rangle$, is in the language of density operators written as $\rho = |\psi\rangle\langle\psi|$, and such states are referred to as "pure", while all other states are referred to as "mixed".

a) For the Pauli operators, defined in exercise 1, show that

- $\mathrm{Tr}(\sigma_j) = 0, \quad j \in \{x, y, z\}$,
- $\mathrm{Tr}(\sigma_j \sigma_k) = 2\delta_{j,k}, \quad j, k \in \{x, y, z\}$.

(4 points)

Remark: The set of linear operators on a finite-dimensional Hilbert space does itself form a Hilbert space (often called the Hilbert-Schmidt space) if one equips it with the inner product $(A, B) = \mathrm{Tr}(A^\dagger B)$. The set of operators $\{\hat{1}, \sigma_x, \sigma_y, \sigma_z\}$ forms an orthogonal (but not normalized) basis of the Hilbert-Schmidt space on a two-dimensional Hilbert space. Via this observation one can understand the Bloch-representation, that is the focus of the next exercise, simply as a basis expansion of vectors.

b) The state of a single spin-half system (a "qubit") can be described via the Bloch-representation

$$\rho = \frac{1}{2}(\hat{1} + \vec{n} \cdot \vec{\sigma}),$$

where $\vec{n} \in \mathbb{R}^3$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Show the following:

- The Bloch-vector is given by $\vec{n} = \mathrm{Tr}(\vec{\sigma}\rho)$
- $\mathrm{Tr}(\rho^2) = \frac{1}{2}(1 + |\vec{n}|^2)$
- $\|\vec{n}\| \leq 1$ for all quantum states, and $\|\vec{n}\| = 1$ for pure states. Hence, the Bloch vectors lie inside a sphere of radius 1, where the pure states lie on the surface.
- Suppose that $\rho_1 = |\psi_1\rangle\langle\psi_1|$ and $\rho_2 = |\psi_2\rangle\langle\psi_2|$ for normalized vectors $|\psi_1\rangle, |\psi_2\rangle$, where these are orthogonal $\langle\psi_1|\psi_2\rangle = 0$. What is the relation between the corresponding Bloch-vectors $\vec{n}_1$ and $\vec{n}_2$?

Hint: Use the fact that $\mathrm{Tr}(\rho^2) \leq 1$ for all density operators, with $\mathrm{Tr}(\rho^2) = 1$ if and only if ρ is pure.

(4 points)