Advanced Quantum Mechanics

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Exercise sheet 3 **Due: Sunday April 23 at 24:00**

1 Tricks for traces

Here we make a few observations that can speed up calculations with traces and partial traces. Recall that when we want to evaluate the trace, we can do it as $\mathrm{Tr}(Q) = \sum_n \langle n|Q|n\rangle$, where $\{|n\rangle\}_n$ is *any* orthonormal basis of $\mathcal{H}$.[1] Also, recall that orthonormal bases satisfy the completeness relation $\sum_n |n\rangle\langle n| = \hat{1}$.

a) Let $|\psi\rangle, |\chi\rangle \in \mathcal{H}$. *Show that*

$$\mathrm{Tr}(|\psi\rangle\langle\chi|) = \langle\chi|\psi\rangle.$$

(1 point)

b) Let $Q : \mathcal{H} \to \mathcal{H}$ be linear, and let $|\psi\rangle \in \mathcal{H}$. *Show that*

$$\langle\psi|Q|\psi\rangle = \mathrm{Tr}(Q|\psi\rangle\langle\psi|).$$

(1 point)

c) Consider two distinguishable quantum systems with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. As we know from the lecture, the Hilbert space of the joint system is $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let $|\psi\rangle, |\phi\rangle \in \mathcal{H}_1$ and $|\alpha\rangle, |\beta\rangle \in \mathcal{H}_2$. *show that*

$$\mathrm{Tr}_2(|\psi\rangle\langle\phi| \otimes |\alpha\rangle\langle\beta|) = |\psi\rangle\langle\phi|\langle\beta|\alpha\rangle$$

(1 point)

d) Suppose that $\{|2_n\rangle\}_n$ is an orthonormal basis of $\mathcal{H}_2$. Suppose that $|\eta\rangle, |\xi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ can be written

$$|\eta\rangle = \sum_n |\alpha_n\rangle|2_n\rangle, \quad |\xi\rangle = \sum_n |\beta_n\rangle|2_n\rangle,$$

where $\{|\alpha_n\rangle\}_n$ and $\{|\beta_n\rangle\}_n$ are sets of (not necessarily orthonormal) elements in $\mathcal{H}_1$. *Express the partial trace* $\mathrm{Tr}_2(|\eta\rangle\langle\xi|)$ *in terms of the α's and β's.* **(1 point)**

2 Two distinguishable systems

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the Hilbert spaces of two distinguishable systems. Suppose that the two systems are in a pure state, where the joint state vector is

$$|\psi\rangle = \frac{1}{\sqrt{3}}|\alpha_1\rangle \otimes |\beta_1\rangle + \sqrt{\frac{2}{3}}|\alpha_2\rangle \otimes |\beta_2\rangle,$$

where $|\alpha_1\rangle, |\alpha_2\rangle$ is an orthonormal set in $\mathcal{H}_1$, and $|\beta_1\rangle, |\beta_2\rangle$ is an orthonormal set in $\mathcal{H}_2$.

[1] If the Hilbert space is infinite-dimensional, then one needs to put restrictions on Q for $\mathrm{Tr}(Q)$ to make sense, but we will not bother about this here. However, if you are curious, you can search for "trace class operators".

a)
- *What is the reduced density operator of particle 1?*
- *Determine whether the reduced state is mixed or pure.*
- *Is the state $|\psi\rangle$ entangled, or is it a product state?*
- *What is the entanglement entropy of $|\psi\rangle$? (Use the base-2 logarithm.)*

(4 points)

b) On system 1, we have the following observable

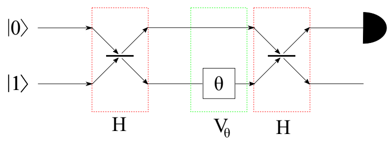
$$Q_1 = 5\big(|\alpha_1\rangle + |\alpha_2\rangle\big)\big(\langle\alpha_1| + \langle\alpha_2|\big) - \frac{1}{2}\big(|\alpha_1\rangle - |\alpha_2\rangle\big)\big(\langle\alpha_1| - \langle\alpha_2|\big).$$

- *Why can we say that Q_1 is an observable?*
- *What is the expectation value of Q_1 with respect to $|\psi\rangle$?*

(3 points)

3 Interferometers and measurements

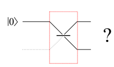
In the lecture we discuss the role of the environment on destroying macroscopic superpositions, as well as a model of measurements. In this exercise we investigate a related type of problem, namely the relation between interference and measurements in single-particle interferometry.



A beam-splitter[2] can put a particle (e.g. a photon) in superposition between propagating in two different paths. We let $|0\rangle$ correspond to the upper path, and $|1\rangle$ the lower path. The ideal beam-splitter can be modeled by the unitary operator[3]

$$H = \frac{1}{\sqrt{2}}|0\rangle\langle 0| + \frac{1}{\sqrt{2}}|1\rangle\langle 0| + \frac{1}{\sqrt{2}}|0\rangle\langle 1| - \frac{1}{\sqrt{2}}|1\rangle\langle 1|.$$

a) Let us consider the first step in the interferometer. Suppose that the particle initially is in path 0, i.e., in state $|0\rangle$. By applying the beam-splitter, we get the state $H|0\rangle$. *Express the state $H|0\rangle$ in terms of the basis states $|0\rangle$ and $|1\rangle$.*

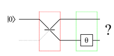


(1 point)

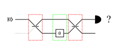
b) Next, we apply a variable phase-shifter, described by the unitary

$$U_\theta = |0\rangle\langle 0| + e^{i\theta}|1\rangle\langle 1|.$$

Express the state $U_\theta H|0\rangle$ in terms of the basis states $|0\rangle$ and $|1\rangle$. **(1 point)**



c) Finally, we let the two paths interfere by applying a second beam-splitter, i.e., the state becomes $HU_\theta H|0\rangle$. *Determine the probability $P(\theta)$ to find the particle in path 0.* **(1 point)**



[2] A beam-splitter could for example be a half-transparent mirror, which lets half of the amplitude pass through, while reflecting half. For single photons, it turns out that this leads to (within a good approximation) that we get a superposition of the photon being reflected or passing through.

[3] It is a bit unfortunate to denote it by H, since this might suggest that it is a Hamiltonian. However, H stands for 'Hardamard gate', and is one of the standard gates in quantum computing.

Remark: With the steps a), b), c), we have constructed a basic two-path interferometer, where the first application of a beam-splitter prepares a superposition between the two paths, the phase-shifter adds a phase-difference between the two paths, and the second beam-splitter generates interference in the measurement.

d) Let us now generalize the setup and replace the initial preparation of the state $H|0\rangle$ in a), with something that instead prepares a general initial state with density operator ρ. If we follow the steps in b) and c), the probability to find the particle in path 0 after the second beam-splitter would be

$$P(\theta) = \langle 0|HU_\theta \rho U_\theta^\dagger H^\dagger|0\rangle = \mathrm{Tr}(|0\rangle\langle 0|HU_\theta \rho U_\theta^\dagger H^\dagger).$$

Show that

$$P(\theta) = \frac{1}{2} + |\langle 1|\rho|0\rangle| \cos\Big(\theta + \arg(\langle 1|\rho|0\rangle)\Big).$$

Check that you regain the result in c) for the special ρ that you constructed in a). **(2 points)**

Remark: The detection-probability P oscillates with θ. One can think of these oscillations as interference-fringes, where the amplitude of these fringes are determined by the magnitude $|\langle 1|\rho|0\rangle|$. One can think of $|\langle 1|\rho|0\rangle|$ as quantifying how 'much' superposition there is between the two paths.[4]

e) By d) we see that it is the superposition between the two paths that gives rise to the interference effect. What would happen with the interference if we would add another system, say a spin, that would be sensitive to the position of the particle? Let us thus add a two-level system, where $|0\rangle_{\text{spin}}$ means 'spin down' and $|1\rangle_{\text{spin}}$ denotes 'spin up'. We also introduce a new unitary operator V that let the particle and the spin interact. We define it as

$$V = \hat{1}_{\text{spin}} \otimes |0\rangle\langle 0| + \sigma_x^{\text{spin}} \otimes |1\rangle\langle 1|,$$

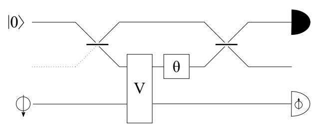
where $\sigma_x^{\text{spin}} = |0\rangle_{\text{spin}}\langle 1| + |1\rangle_{\text{spin}}\langle 0|$ is the Pauli-x operator on the spin.[5] As one can see, nothing happens to the spin if the particle is in path 0, while the spin flips if the particle is in path 1. *Express $VH|0\rangle_{\text{spin}}|0\rangle$ in terms of the basis elements $|0\rangle_{\text{spin}}|0\rangle$, $|0\rangle_{\text{spin}}|1\rangle$, $|1\rangle_{\text{spin}}|0\rangle$, $|1\rangle_{\text{spin}}|1\rangle$. Show that it is maximally entangled, and that the spin is in the $|0\rangle_{\text{spin}}$-state whenever the particle is in path 0, and that it is in the $|1\rangle_{\text{spin}}$-state whenever the particle is in path 1, i.e., that they are perfectly correlated.*

(2 points)

Remark: One can think of the spin as measuring the position of the particle.

f) Suppose that we now apply HU_θ (like in b) and c)) on $VH|0\rangle_{\text{spin}}|0\rangle$. *Would one see any interference in this new setup?*

(2 points)

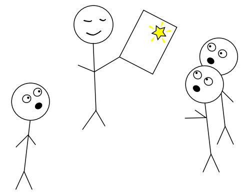


[4] The density operator ρ is represented in the basis $\{|0\rangle, |1\rangle\}$ by the matrix $\begin{bmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{bmatrix}$. The diagonal elements $\langle 0|\rho|0\rangle$ and $\langle 1|\rho|1\rangle$ give the probability to find the system in states $|0\rangle$ and $|1\rangle$, respectively. By the interferometric setup, we also get in interpretation of the off-diagonal elements $\langle 1|\rho|0\rangle$ and $\langle 0|\rho|1\rangle$ as telling us how strongly superposed the two states are, in the sense that $|\langle 1|\rho|0\rangle|$ gives the magnitude of the interference.

[5] V is an example of a CNOT gate (controlled-not gate).

4 Gold star exercise: A bit more on the interferometer

This exercise gives absolutely no points! However, by solving it you do of course become the center of admiration and envy for all your peers. Moreover, your family will (obviously) bask in unbound glory for at least three generations.



Here we continue further into the investigation of the interferometer in problem 3. In problem 3 e) we considered a specific unitary operator V that describes the interaction between the spin and the particle. Here we consider a more general class of such unitary operators

$$V_\chi = \hat{1}_{\text{spin}} \otimes |0\rangle\langle 0| + [i\cos(\chi)\hat{1} + \sin(\chi)\sigma_x] \otimes |1\rangle\langle 1|$$

where the parameter χ determines how sensitive the spin is to where the particle is.

a) *Express the state $V_\chi H|0\rangle_{\text{spin}}|0\rangle$ in terms of the basis elements $|0\rangle_{\text{spin}}|0\rangle$, $|0\rangle_{\text{spin}}|1\rangle$, $|1\rangle_{\text{spin}}|0\rangle$, $|1\rangle_{\text{spin}}|1\rangle$.* In the case $\chi = \pi/2$ we regain problem 3 e). *In the case $\chi = 0$, does the state of the spin depends on where the particle is?*

(0 points)

b) As a simple measure of how coordinated that the spin and the path is, consider the expectation-value of the observable[6]

$$Q = |0\rangle_{\text{spin}}\langle 0| \otimes |0\rangle\langle 0| + |1\rangle_{\text{spin}}\langle 1| \otimes |1\rangle\langle 1|.$$

Evaluate the expectation value

$$P_{\text{spin}}(\chi) = \langle 0, 0_{\text{spin}}|H^\dagger V_\chi^\dagger Q V_\chi H|0_{\text{spin}}, 0\rangle.$$

(0 points)

Remark: The idea is that when $P_{\text{spin}}(\chi)$ is large, then the spin is very sensitive to the location of the particle, while when it is low, then the spin does not notice much of the location.

c) *Apply the steps in 3b) and 3c) to the state $V_\chi H|0\rangle_{\text{spin}}|0\rangle$, and determine the magnitude of the interference fringes as a function of χ How does the detection probability $P_{\text{spin}}(\chi)$ in b) relates to the magnitude of the interference fringes. Can they both be large at the same time?*

Hint: Determine the reduced density operator of the particle, and use problem 3d).

(0 points)

[6]This is not a particularly good measure of correlations, but it is simple to calculate.