Advanced Quantum Mechanics

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Exercise sheet 4 **Due: Sunday April 30 at 24:00**

1 Gates and circuits

a) The NOT-gate, or X-gate[1], is a single-qubit gate that acts like $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$. With respect to the computational basis $|0\rangle, |1\rangle$, it has the matrix representation $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. We can also write it directly in terms of the basis as

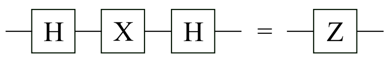
$$X = |1\rangle\langle 0| + |0\rangle\langle 1|. \tag{1}$$

Similarly, the Z-gate is a single-qubit gate that acts like $Z|x\rangle = (-1)^x|x\rangle$, and the Y-gate acts like $Y|0\rangle = i|1\rangle$ and $Y|1\rangle = -i|0\rangle$. We also have the Hadamard gate that acts like $H|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle)$. *Analogous to (1), express Z, Y and H in terms of the computational basis elements.*

Hint: Resolutions of identity can often be useful.

(3 points)

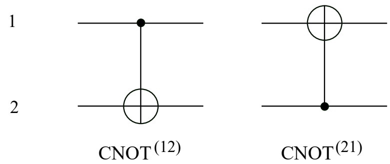
b) *Show the following equivalence*



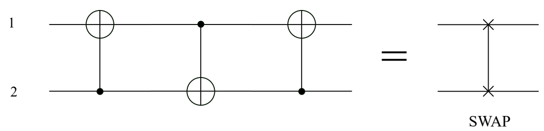
(1 point)

c) Recall that the CNOT-gate swaps the state of a target-bit depending on the state of a control-bit. For two bits, this gate thus comes in two versions, depending on which bit is the control, and which is the target. With respect to the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ the two corresponding gates are (where we use the ordering $|x_1x_2\rangle$)

$$CNOT^{(12)}|00\rangle = |00\rangle, \quad CNOT^{(12)}|01\rangle = |01\rangle, \quad CNOT^{(12)}|10\rangle = |11\rangle, \quad CNOT^{(12)}|11\rangle = |10\rangle,$$
$$CNOT^{(21)}|00\rangle = |00\rangle, \quad CNOT^{(21)}|01\rangle = |11\rangle, \quad CNOT^{(21)}|10\rangle = |10\rangle, \quad CNOT^{(21)}|11\rangle = |01\rangle,$$



If one concatenates these two types of CNOT-gates such that $CNOT^{(21)}CNOT^{(12)}CNOT^{(21)}$, one obtains a gate referred to as the SWAP-gate.



[1] The NOT-gate is often written as the X-gate in the quantum-case, since it corresponds to the Pauli-x operator, analogous to how the Z-gate and Y-gate correspond to the Pauli-z and -y operators.

Show that the SWAP-gate has the property that

$$SWAP|\alpha\rangle|\beta\rangle = |\beta\rangle|\alpha\rangle,$$

for any input state $|\alpha\rangle$ on the first qubit and any input state $|\beta\rangle$ on the second qubit. In other words, the SWAP-gate swaps the states of qubits 1 and 2.

(3 points)

d) *Show that*

$$CNOT^{(12)} = |0\rangle\langle 0| \otimes \hat{1}_2 + |1\rangle\langle 1| \otimes X.$$

(1 point)

e) *Find a two-qubit Hamiltonian that implements the $CNOT^{(12)}$-gate.* It is fine if the implementation agrees with the $CNOT^{(12)}$-gate only up to a global phase-factor.

Hint: There was a reason for why we did d).

(2 points)

f) In analogy with the CNOT-gate, one can consider all kinds of controlled gates on the form $G^{(12)} = |0\rangle\langle 0| \otimes \hat{1}_2 + |1\rangle\langle 1| \otimes U$, some single-qubit unitary operator U on the second qubit. We might for example want to implement control-Z-gate in this manner, with $U = Z$. *Suppose that you have access to $CNOT^{(12)}$-gates and Hadamard gates. How could you use these in order to implement a control-Z-gate? Draw the circuit that would implement the control-Z-gate.*

Hint: Again, there was a reason for why we did d).

(2 points)

2 Showing that U_f is unitary

For the analysis of Grover's algorithm, we did in the lecture model the computational problem via a function $f : \{0,1\}^{\times n} \to \{0,1\}$, which outputs 1 if the n-bit string x is a solution to the problem, while it otherwise outputs 0. To turn this into a quantum gate, we consider a n-qubit system with with computational basis $\{|x\rangle\}_{x\in\{0,1\}^{\times n}}$.[2] Moreover, we introduce an extra qubit with basis $\{|y\rangle\}_{y=0,1}$, and define the operator U_f by its action on all the basis elements $|x\rangle|y\rangle$ as

$$U_f|x,y\rangle = |x, f(x) \oplus y\rangle, \quad x \in \{0,1\}^{\times n}, \quad y \in \{0,1\}. \tag{2}$$

Here $\oplus$ is addition modulo 2.[3] *Show that U_f is a unitary operator.*

(4 points)

3 From U_f to V_f

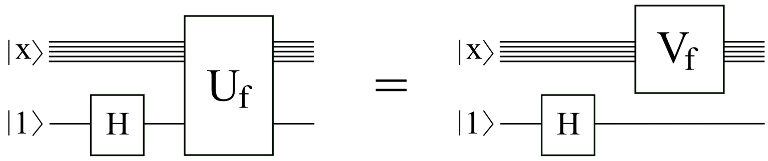
In the lecture we made a jump from U_f in (2) to the operator V_f such that

$$V_f|x\rangle = (-1)^{f(x)}|x\rangle, \quad x \in \{0,1\}^{\times n}.$$

[2] When one talks about a 'computational basis' it is implied that we consider a product basis $|x\rangle = |x_1\rangle \otimes \cdots \otimes |x_n\rangle$ with a 'local' basis on each qubit. This is good to keep in mind, since there are many other choices of basis that are not of this product form.

[3] The truth-table of $a \oplus b$ for two single bits a and b is $\begin{bmatrix} a & b & a\oplus b \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Hence, this is equivalent to $XOR(a,b)$.

(Hence, as opposed to U_f in (2), this does not require any 'extra' qubit.) In the lecture we claimed that if we have access to a device that implements U_f, then we can also implement V_f. In this exercise we show how this can be done. *Verify the equivalence depicted in the circuit diagrams below.* (Recall that circuit diagrams are read from left to right.) In other words, if we first put the extra qubit (corresponding to y in (2)) into the state $H|1\rangle$, before we apply U_f, then the effect on $|x\rangle$ is as if we applied U_f and did do nothing on $H|1\rangle$.[4]

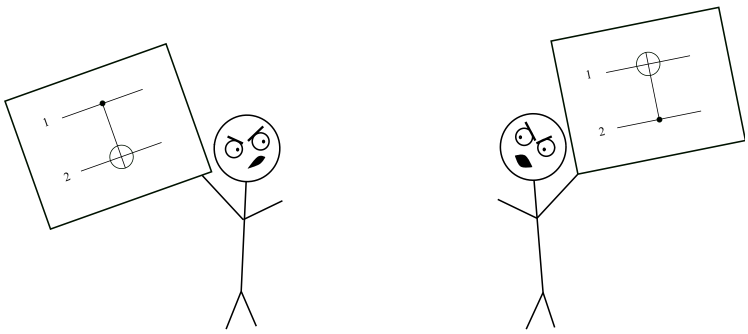


Hint: Might ease things if one treats different cases separately.

(4 points)

4 Gold star exercise: Alice and Bob fall out over the CNOT-gate

This exercise gives absolutely no points, but you can do it anyway if you want to.



Students Alice and Bob[5] have registered a startup "Cologne Quantum Supercomputing" and built a two-qubit quantum computer in their WG.[6] They believe that they have implemented the CNOT-gate, but decided to test it experimentally. The problem is that they cannot agree on the results, and they are getting increasingly angry at each other.

Alice and Bob each performs a series of experiments. Alice uses the computational basis $|0^A0^A\rangle$, $|0^A1^A\rangle$, $|1^A0^A\rangle$, $|1^A1^A\rangle$ and finds that G operates as

$$G|0_1^A0_2^A\rangle = |0_1^A0_2^A\rangle, \quad G|0_1^A1_2^A\rangle = |0_1^A1_2^A\rangle, \quad G|1_1^A0_2^A\rangle = |1_1^A1_2^A\rangle, \quad G|1_1^A1_2^A\rangle = |1_1^A0_2^A\rangle.$$

In other words, G acts precisely as one would expect from the $CNOT^{(12)}$-gate. Bob happens to choose a different computational basis $|0_1^B0_2^B\rangle$, $|0_1^B1_2^B\rangle$, $|1_1^B0_2^B\rangle$, $|1_1^B1_2^B\rangle$, which is related to Alice's basis as $|x_1^Bx_2^B\rangle = H \otimes H|x_1^Ax_2^A\rangle$. The problem is that when Bob performs his experiment, he finds that

$$G|0_1^B0_2^B\rangle = |0_1^B0_2^B\rangle, \quad G|0_1^B1_2^B\rangle = |1_1^B1_2^B\rangle, \quad G|1_1^B0_2^B\rangle = |1_1^B0_2^B\rangle, \quad G|1_1^B1_2^B\rangle = |0_1^B1_2^B\rangle,$$

[4] This is maybe a bit counter-intuitive. The extra qubit is necessary in order implement V_f on $|x\rangle$, but the state of the extra-qubit itself is left unaffected at the end of the process. Because of this, when we analyze Grover's algorithm, it is enough to consider V_f without having to bother about the extra qubit.

[5] You may have known them under the names Anna and Bernd.

[6] The equipment occupies 93% of the volume of their flat. They only have space to sleep in shifts, and they can only barely squeeze themselves between the kitchen and the bathroom. There is constant noise from the cooling equipment, and the floor seems to bulge suspiciously under the weight, but who cares about such minor inconveniences!? They intend to sell access to their quantum computer via a cloud service and become insanely rich!!!

which is precisely what you would expect from a $CNOT^{(21)}$-gate. When they repeat their experiments, they find the same results over and over again, and each is getting more and more convinced that the other is completely incompetent and a total moron. Hopefully, you can help to resolve their issue.

a) Let us start with some general observations. The classical $CNOT^{(12)}$-gate flips the second bit x_2 only if the first bit is $x_1 = 1$. Moreover, it leaves the first bit unaffected. The analogous statements are, by construction, true for the corresponding quantum gate, if it is applied to the computational basis states. What happens if we apply the quantum $CNOT^{(12)}$-gate to more general states, e.g., to product states $|\psi_1\rangle|\psi_2\rangle$? (There is no superscript A or B on purpose, in order to leave it open.) Let us simply try! *Apply $CNOT^{(12)}$ on the product state $\frac{1}{\sqrt{2}}(|0_1\rangle + |1_1\rangle)|0_2\rangle$. Is the output a product state? Is the state of qubit 1 unaffected?*

(0 points)

b) *Show that*

$$H^{\otimes 2} CNOT^{(12)} H^{\otimes 2} = CNOT^{(21)}.$$

(0 points)

c) Suppose that Alice defines the unitary operator $CNOT^{(12),A}$ via $CNOT^{(12),A}|x_1^A x_2^A\rangle = |y_1^A y_2^A\rangle$ with $(y_1, y_2) = CNOT^{(12)}(x_1, x_2)$ for her basis $\{|x_1^A x_2^A\rangle\}_{x_1,x_2}$. Suppose moreover that Bob defines the unitary operator $CNOT^{(21),B}$ via $CNOT^{(21),B}|x_1^B x_2^B\rangle = |y_1^B y_2^B\rangle$ with $(y_1, y_2) = CNOT^{(12)}(x_1, x_2)$ for his basis $\{|x_1^B x_2^B\rangle\}_{x_1,x_2}$. *What is the relation between the operators $CNOT^{(12),A}$ and $CNOT^{(21),B}$? Do you have any suggestion for how Alice and Bob's friendship could be saved?*

(0 points)

Moral of this story: The quantum counterparts of classical gates are not as innocent as they might seem at first sight; they might look quite different in another computational basis. Moreover, the quantum counterpart to a classical gate is always defined with respect to a *choice* of computational basis, so the actual operator depends on which basis that you happen to choose.