Advanced Quantum Mechanics

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SS 2023

Exercise sheet 5 **Due: Sunday May 7 at 24:00**

1 Symmetric and anti-symmetric subspaces for two distinguishable particles

In the lecture we defined the exchange operator[1] τ_{kl} via its action on orthonormal n-body basis elements $\tau_{kl}|\ldots,i_k,\ldots,i_l,\ldots\rangle = |\ldots,i_l,\ldots,i_k,\ldots\rangle$.

Suppose that we have two *distinguishable* particles, each with Hilbert space $\mathcal{H}$. The total Hilbert space of the two particles is $\mathcal{H}\otimes\mathcal{H}$ and we have the orthonormal product basis $|i_1,i_2\rangle = |i_1\rangle\otimes|i_2\rangle$. In this case there is thus only one single exchange-operator τ_{12}, which acts like

$$\tau_{12}(|i_1\rangle\otimes|i_2\rangle) = |i_2\rangle\otimes|i_1\rangle. \tag{1}$$

We say that a state $|\xi\rangle \in \mathcal{H}\otimes\mathcal{H}$ is symmetric if $\tau_{12}|\xi\rangle = |\xi\rangle$, and that it is anti-symmetric if $\tau_{12}|\xi\rangle = -|\xi\rangle$.

Remark: There was a discussion on Slack last Thursday (27/4) about $\mathbb{C}^n\otimes\mathbb{C}^n$. This exercise is essentially about that discussion.

a) *Based on the definition (1), show that τ_{12} can be written*

$$\tau_{12} = \sum_{i_1,i_2} |i_2\rangle\langle i_1|\otimes|i_1\rangle\langle i_2|.$$

(2 points)

b) *Show that τ_{12} is Hermitian, unitary, and satisfies $\tau_{12}^2 = \hat{1}_{\mathcal{H}}\otimes\hat{1}_{\mathcal{H}}$.*

Hint: There was a reason for why we did a). **(2 points)**

c) *Show that $\tau_{12}|\psi\rangle\otimes|\chi\rangle = |\chi\rangle\otimes|\psi\rangle$.* **(1 point)**

Remark: This suggests that τ_{12} indeed swaps the states of the two particles.

d) Define the two operators $P_S = \frac{1}{2}(\hat{1}+\tau_{12})$ and $P_A = \frac{1}{2}(\hat{1}-\tau_{12})$.
Show that P_S and P_A are projectors.[2] **(2 points)**

Remark: Recall that to every projector corresponds a subspace, and vice versa. Hence, P_S corresponds to a subspace $\mathcal{L}_S\subset\mathcal{H}\otimes\mathcal{H}$, and P_A corresponds to a subspace $\mathcal{L}_A\subset\mathcal{H}\otimes\mathcal{H}$.

e) *Show that the projectors P_S and P_A are orthogonal to each other, in the sense that $P_SP_A = P_AP_S = 0$. Show also that P_S and P_A are complementary, in the sense that $P_S + P_A = \hat{1}$.* **(2 points)**

Remark: We conclude that every element of $\mathcal{H}\otimes\mathcal{H}$ can be decomposed as an orthogonal sum of one element in $\mathcal{L}_S$ and one element in $\mathcal{L}_A$. In other words, $\mathcal{L}_S$ is orthogonal to $\mathcal{L}_A$, and $\mathcal{L}_S\oplus\mathcal{L}_A = \mathcal{H}\otimes\mathcal{H}$.

[1] Also referred to as the permutation operator, or the swap operator. In the special case of $N = 2$, recall the swap-gate that we introduced in sheet 4.

[2] What we here refer to as a "projector", a mathematician typically would refer to as an "orthogonal projector".

f) *Show that $\tau_{12}P_S = P_S$ and that $\tau_{12}P_A = -P_A$.* **(1 point)**

Remark: This means that all elements in $\mathcal{L}_S$ are symmetric, while all elements of $\mathcal{L}_A$ are anti-symmetric. This also means that if $|\xi\rangle$ is any element in $\mathcal{H}\otimes\mathcal{H}$, then we know that $P_S|\xi\rangle$ is symmetric, and $P_A|\xi\rangle$ is anti-symmetric. Hence, this is a method to construct symmetric or anti-symmetric states.

Remark on the whole exercise: We have discovered that the space of two distinguishable particles (with isomorphic Hilbert spaces) decomposes into the symmetric subspace $\mathcal{L}_S$ and the anti-symmetric subspace $\mathcal{L}_A$. Note that $\mathcal{L}_S$ would be the state space if the two particles were identical bosons, and $\mathcal{L}_A$ would be the state space is they would be identical fermions. Hence, the Hilbert space of the two distinguishable particles decomposes into that of two bosons and that of two fermions. However, as mentioned in that discussion on Slack, this nice picture does unfortunately not survive if we have more than two particles, since more subspaces pop up in this case. In case you want to read further about these things, these additional subspaces are related to the notion of parastatistics.

2 The social life of fermions and bosons

Suppose that $|\psi_a\rangle$ and $|\psi_b\rangle$ are two orthonormal single-particle states. One possible two-particle state for two *distinguishable* particles would be $|\psi_{\text{dist}}\rangle = |\psi_a\rangle_1|\psi_b\rangle_2$, meaning that particle 1 is in state ψ_a and particle 2 is in state ψ_b. Let X_1 and X_2 be the position operator of particle 1 and 2, respectively. The operator $(X_1 - X_2)^2$ (or $(X_1\otimes\hat{1}_2 - \hat{1}_1\otimes X_2)^2$ in a more elaborate notation) would thus quantify the (square) distance between the two particles. Let $|\psi_{\text{Bose}}\rangle = \frac{1}{\sqrt{2}}(|\psi_a\rangle_1|\psi_b\rangle_2 + |\psi_b\rangle_1|\psi_a\rangle_2)$ be the two-particle symmetrized state, and let $|\psi_{\text{Fermi}}\rangle = \frac{1}{\sqrt{2}}(|\psi_a\rangle_1|\psi_b\rangle_2 - |\psi_b\rangle_1|\psi_a\rangle_2)$ be the two-particle anti-symmetrized state.

a) Write $\langle\psi_{\text{dist}}|(X_1 - X_2)^2|\psi_{\text{dist}}\rangle$ in terms of the quantities[3]

$$\begin{aligned}\Delta^2_{\psi_a} &= \langle\psi_a|(X - \langle\psi_a|X|\psi_a\rangle)^2|\psi_a\rangle, \quad \Delta^2_{\psi_b} = \langle\psi_b|(X - \langle\psi_b|X|\psi_b\rangle)^2|\psi_b\rangle,\\ D^2_{\psi_a,\psi_b} &= (\langle\psi_a|X|\psi_a\rangle - \langle\psi_b|X|\psi_b\rangle)^2.\end{aligned}$$

(2 points)

b) *Do the same for $\langle\psi_{Bose}|(X_1 - X_2)^2|\psi_{Bose}\rangle$.*

Hint: Apart from $\Delta^2_{\psi_a}$, $\Delta^2_{\psi_b}$, D^2, a new term appears.

(2 points)

c) *Analogous to a) and b), determine $\langle\psi_{Fermi}|(X_1 - X_2)^2|\psi_{Fermi}\rangle$. It is often stated that bosons are 'social' and tend to cluster, while fermions tend to avoid each other. How do the above results fit with these claims?*

Hint: Note that $\Delta^2_{\psi_a} \geq 0$, $\Delta^2_{\psi_b} \geq 0$ and $D^2_{\psi_a,\psi_b} \geq 0$.

(2 points)

[3] The quantities $\Delta^2_{\psi_a}$ and $\Delta^2_{\psi_b}$ are simply the variances of the single-particle states ψ_a and ψ_b (with respect to position). The quantity $D^2_{\psi_a,\psi_b}$ quantifies the size of the difference between the average positions of the two particles.

3 Labeling of particles versus the occupation number representation

So far we have treated identical particles by labeling them, and followed the rule that the state-vector has to be totally symmetric or totally anti-symmetric depending on whether we have bosons or fermions. In the occupation number representation (or Fock representation) we do instead specify how a collection of (orthogonal) single-particle states are occupied. This method of describing the system does not involve any labeling of particles, and there is no need for symmetrization or anti-symmetrization. As you most likely will notice, this method tends to give a more compact notation. In the following, let $|\phi_\alpha\rangle$, $|\phi_\beta\rangle$, $|\phi_\gamma\rangle$, $|\phi_\delta\rangle$, and $|\phi_\epsilon\rangle$ be a collection of (orthonormal) single particle state-vectors.

a) Let $|0{,}2{,}1{,}0{,}1\rangle$ be a bosonic occupation number state-vector with respect to the ordering of the single-particle states indicated above. (Hence, there are two quanta in state ϕ_β, one in ϕ_γ and one in ϕ_ϵ.) *Determine the multi-particle wave-function* $\psi(x_1, x_2, x_3, x_4) = \langle x_1, x_2, x_3, x_4|0{,}2{,}1{,}0{,}1\rangle$ *in terms of the wave functions* $\phi_\alpha(x) = \langle x|\phi_\alpha\rangle$, $\phi_\beta(x) = \langle x|\phi_\beta\rangle$, *etc.*

(2 points)

b) Consider the following totally anti-symmetric state-vector over three particles

$$
\begin{aligned}
|\psi_a\rangle = \frac{1}{\sqrt{6}}\Big(&|\phi_\alpha\rangle_1|\phi_\beta\rangle_2|\phi_\delta\rangle_3 + |\phi_\delta\rangle_1|\phi_\alpha\rangle_2|\phi_\beta\rangle_3 + |\phi_\beta\rangle_1|\phi_\delta\rangle_2|\phi_\alpha\rangle_3 \\
&- |\phi_\beta\rangle_1|\phi_\alpha\rangle_2|\phi_\delta\rangle_3 - |\phi_\delta\rangle_1|\phi_\beta\rangle_2|\phi_\alpha\rangle_3 - |\phi_\alpha\rangle_1|\phi_\delta\rangle_2|\phi_\beta\rangle_3\Big).
\end{aligned}
$$

Write down the corresponding fermionic occupation number representation, with respect to the ordering of the single-particle states indicated above.

(2 points)