Advanced Quantum Mechanics

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Exercise sheet 8 **Due: Sunday May 28 at 24:00**

1 Conservation of particle-number for one- and two-body Hamiltonians

For a finite (or countable) number of modes, Hamiltonians that contain single-particle terms, as well as two-particle terms, can be written on the form

$$H = \sum_{jk} T_{jk} a_j^\dagger a_k + \sum_{jkmn} F_{jkmn} a_j^\dagger a_k^\dagger a_n a_m.$$

In this exercise we will show that Hamiltonians of this form always commute with the total number operator, in both the bosonic and fermionic case. This means that the particle number is conserved, i.e., the dynamics induced by H does not change the total number of particles in the system.

Recall that a collection of bosonic annihilation and creation operators a_k, $a_k^\dagger$ satisfy the bosonic commutation relations

$$[a_k, a_{k'}] = 0, \quad [a_k^\dagger, a_{k'}^\dagger] = 0, \quad [a_k, a_{k'}^\dagger] = \delta_{k,k'} \hat{1},$$

while a collection of fermionic annihilation and creation operators satisfy the anti-commutator relations

$$\{a_k, a_{k'}\} = 0, \quad \{a_k^\dagger, a_{k'}^\dagger\} = 0, \quad \{a_k, a_{k'}^\dagger\} = \delta_{k,k'} \hat{1},$$

where $\{A, B\} = AB + BA$ is the anti-commutator.

a) If $a_j, a_j^\dagger$ are bosonic annihilation and creation operators, *show that*

$$\begin{aligned}
[a_j^\dagger a_k, a_l^\dagger a_l] =& a_j^\dagger a_l \delta_{k,l} - a_l^\dagger a_k \delta_{l,j}, \\
[a_j^\dagger a_k^\dagger a_n a_m, a_l^\dagger a_l] =& a_j^\dagger a_k^\dagger a_n a_l \delta_{m,l} + a_j^\dagger a_k^\dagger a_m a_l \delta_{n,l} - a_l^\dagger a_j^\dagger a_n a_m \delta_{l,k} - a_l^\dagger a_k^\dagger a_n a_m \delta_{l,j}
\end{aligned}$$

Hint: Recall that $[AB, C] = A[B, C] + [A, C]B$ and $[A, BC] = B[A, C] + [A, B]C$, and then apply the bosonic commutation relations.

(3 points)

b) *Show that the total number operator $N = \sum_l a_l^\dagger a_l$ commutes with H in the bosonic case.* **(2 points)**

c) If $a_j, a_j^\dagger$ are fermionic annihilation and creation operators, *show that*

$$\begin{aligned}
[a_j^\dagger a_k, a_l^\dagger a_l] =& a_j^\dagger a_l \delta_{k,l} - a_l^\dagger a_k \delta_{l,j}, \\
[a_j^\dagger a_k^\dagger a_n a_m, a_l^\dagger a_l] =& a_j^\dagger a_k^\dagger a_n a_l \delta_{m,l} - a_j^\dagger a_k^\dagger a_m a_l \delta_{n,l} + a_l^\dagger a_j^\dagger a_n a_m \delta_{l,k} - a_l^\dagger a_k^\dagger a_n a_m \delta_{l,j}.
\end{aligned}$$

Hint: Again one can make use of $[AB, C] = A[B, C] + [A, C]B$ and $[A, BC] = B[A, C] + [A, B]C$. Next, one can show that $[AB, C] = A\{B, C\} - \{A, C\}B$. Finally, one can use the fermionic anti-commutation relations.

(3 points)

d) *Show that the total number operator $N = \sum_l a_l^\dagger a_l$ commutes with H also in the fermionic case.*

(2 points)

2 Some steps in the Bose-gas derivation

In the lecture (section 2.6 in the lecture notes) we analyze a Bose gas, and to each momentum $\vec{k}$ we associate corresponding bosonian annihilation and creation operators $a_{\vec{k}}$ and $a^{\dagger}_{\vec{k}}$. After a bit of juggling, we define a new set of annihilation and creation operators $c_{\vec{k}}$, $c^{\dagger}_{\vec{k}}$ by

$$\begin{aligned} c_{\vec{k}} &= u_{\vec{k}} a_{\vec{k}} - v_{\vec{k}} a^{\dagger}_{-\vec{k}} \quad , \quad u_{\vec{k}} = \frac{1}{2}\left(\sqrt{\frac{\epsilon_{\vec{k}}}{E_{\vec{k}}}} + \sqrt{\frac{E_{\vec{k}}}{\epsilon_{\vec{k}}}}\right), \quad v_{\vec{k}} = \frac{1}{2}\left(\sqrt{\frac{\epsilon_{\vec{k}}}{E_{\vec{k}}}} - \sqrt{\frac{E_{\vec{k}}}{\epsilon_{\vec{k}}}}\right), \\ \epsilon_{\vec{k}} &= \frac{\hbar^2 \|\vec{k}\|^2}{2m}, \quad E_{\vec{k}} = \sqrt{\epsilon^2_{\vec{k}} + \epsilon_{\vec{k}} 2U\rho}, \end{aligned} \tag{1}$$

where we can think of $c_{\vec{k}}$ and $c^{\dagger}_{\vec{k}}$ as annihilating and creating quasi-particles. In this exercise we are going to check a couple of the claims in the lecture notes concerning these quasi-particles.

a) *Confirm that $c_{\vec{k}}$, $c^{\dagger}_{\vec{k}}$ really are bosonic annihilation and creation operators.*

(2 points)

b) *Show that (1) implies*

$$a_{\vec{k}} = u_{\vec{k}} c_{\vec{k}} + v_{\vec{k}} c^{\dagger}_{-\vec{k}}.$$

Hint: How does $\epsilon_{-\vec{k}}$ relate to $\epsilon_{\vec{k}}$? How do $u_{-\vec{k}}$ and $v_{-\vec{k}}$ relate to $u_{\vec{k}}$ and $v_{\vec{k}}$? In the lecture notes there is a relation between $u^2_{\vec{k}}$ and $v^2_{\vec{k}}$ that could be useful.

(2 points)

c) The momentum operator $\vec{P}$ can, with respect to the annihilation and creation operators $a_{\vec{k}}$ and $a^{\dagger}_{\vec{k}}$, be written as

$$\vec{P} = \hbar \sum_{\vec{l}} \vec{l} a^{\dagger}_{\vec{l}} a_{\vec{l}}.$$

Show that

$$\vec{P} = \hbar \sum_{\vec{l}} \vec{l} c^{\dagger}_{\vec{l}} c_{\vec{l}}.$$

Hint: There was a reason for why we bothered to do b). In the sum over $\vec{l}$, we could equally well sum over $-\vec{l}$. Are there some terms that are anti-symmetric under the swap of $\vec{l}$ to $-\vec{l}$?

(4 points)

d) To the quasi-particles we associate a vacuum state $|0\rangle^{(q)}$ by the requirement that it should be annihilated by all the annihilation operators, i.e., $c_{\vec{k}}|0\rangle^{(q)} = 0$ for all $\vec{k}$. (Not to be confused with the vacuum state $|0\rangle^{(p)}$ defined by $a_{\vec{k}}|0\rangle^{(p)} = 0$ for all $\vec{k}$.) In the lecture notes it is claimed that $c^{\dagger}_{\vec{k}}$ creates (when applied to $|0\rangle^{(q)}$) a state with momentum $\hbar\vec{k}$. *Show that $c^{\dagger}_{\vec{k}}|0\rangle^{(q)}$ is an eigenstate of the momentum operator $\vec{P}$, with corresponding eigenvalue $\hbar\vec{k}$.*[1]

(2 points)

3 Gold star exercise: Bogoliubov transformations

In exercise 1b) on sheet 7, we considered Bogoliubov transformations on one single bosonic mode. However, this can be generalized to several modes. The mapping given by $c_{\vec{k}} = u_{\vec{k}} a_{\vec{k}} - v_{\vec{k}} a^{\dagger}_{-\vec{k}}$ in exercise 2 is an example of a Bogoliubov transformation involving two modes.

Given a collection of bosonic annihilation operators $\{a_k\}_{k=1}^{K}$ and creation operators $\{a_k^{\dagger}\}_{k=1}^{K}$. Let us now define a new collection of operators $\{b_k\}_{k=1}^{K}$ by

$$b_k = \sum_l A_{k,l} a_l + \sum_l B_{k,l} a_l^{\dagger} \tag{2}$$

and consequently

$$b_k^{\dagger} = \sum_l A_{k,l}^* a_l^{\dagger} + \sum_l B_{k,l}^* a_l, \tag{3}$$

where $\boldsymbol{A} = [A_{k,l}]_{k,l}$ and $\boldsymbol{B} = [B_{k,l}]_{k,l}$ are complex $K \times K$ matrices.

a) *Determine the necessary and sufficient conditions on $\boldsymbol{A}$ and $\boldsymbol{B}$ for $\{b_k\}_{k=1}^{K}$ and $\{b_k^{\dagger}\}_{k=1}^{K}$ to be bosonic annihilation and creation operators.*

(0 points)

b) The mappings (2) and (3) can be summarized as

$$\begin{bmatrix} b_1 \\ \vdots \\ b_K \\ b_1^{\dagger} \\ \vdots \\ b_K^{\dagger} \end{bmatrix} = \boldsymbol{C} \begin{bmatrix} a_1 \\ \vdots \\ a_K \\ a_1^{\dagger} \\ \vdots \\ a_K^{\dagger} \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^* & \boldsymbol{A}^* \end{bmatrix},$$

where $\boldsymbol{A}^*$ means that we take the complex conjugate of each component of the matrix. With $\boldsymbol{I}$ the $K \times K$ identity matrix, let us define

$$\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{0} & -\boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}.$$

Show that the conditions that you found in a) can be rephrased as

$$\boldsymbol{C} \boldsymbol{\Omega} \boldsymbol{C}^t = \boldsymbol{\Omega}, \tag{4}$$

where $\boldsymbol{C}^t$ denotes the transpose of $\boldsymbol{C}$.

(0 points)

Remark: As you potentially may know from classical mechanics, a matrix $\boldsymbol{C}$ that satisfies (4) is referred to as being symplectic. It is no coincidence that this notion reappears in quantum mechanics.

c) Looking at problem 2, the relation $c_{\vec{k}} = u_{\vec{k}} a_{\vec{k}} - v_{\vec{k}} a^{\dagger}_{-\vec{k}}$ induces a mapping from $a_{\vec{k}}, a_{-\vec{k}}, a^{\dagger}_{\vec{k}}, a^{\dagger}_{-\vec{k}}$ to $c_{\vec{k}}, c_{-\vec{k}}, c^{\dagger}_{\vec{k}}, c^{\dagger}_{-\vec{k}}$. *What is the corresponding matrix $\boldsymbol{C}$? Confirm that $\boldsymbol{C}$ is symplectic.*

(0 points)

[1] Since $\vec{P}$ is a vector-operator, the eigenvalue $\hbar\vec{k}$ is a vector, and the eigenvalue equation reads $\vec{P} c_{\vec{k}} |0\rangle^{(q)} = \hbar\vec{k} c_{\vec{k}} |0\rangle^{(q)}$. In terms of the components $\vec{P} = (P_1, P_2, P_3)$ and $\vec{k} = (k_1, k_2, k_3)$, we thus have three simultaneous eigenvalue equations $P_1 c^{\dagger}_{\vec{k}} |0\rangle^{(q)} = \hbar k_1 c^{\dagger}_{\vec{k}} |0\rangle^{(q)}$, $P_2 c^{\dagger}_{\vec{k}} |0\rangle^{(q)} = \hbar k_2 c^{\dagger}_{\vec{k}} |0\rangle^{(q)}$ and $P_3 c^{\dagger}_{\vec{k}} |0\rangle^{(q)} = \hbar k_3 c^{\dagger}_{\vec{k}} |0\rangle^{(q)}$.