Advanced Quantum Mechanics

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Exercise sheet 9 **Due: Sunday June 11 at 24:00**

1 Energy and momentum of the electromagnetic field

As a preparation for the quantization of the electromagnetic field (Section 3.2 in the lecture notes) we introduce the energy of the electromagnetic field

$$H_{\text{em}} = \frac{\epsilon_0}{2}\int \left(\|\boldsymbol{E}(t,\boldsymbol{x})\|^2 + c^2\|\boldsymbol{B}(t,\boldsymbol{x})\|^2\right) d^3x. \tag{1}$$

Note that we have not yet quantized the field, and we thus regard $a_{\boldsymbol{k},\lambda}$ as complex numbers.

For the calculations, the electric and magnetic fields can be written as

$$\begin{aligned}
\boldsymbol{E}(t,\boldsymbol{x}) =& i\sqrt{\frac{\hbar}{\epsilon_0 L^3}}\sum_{\boldsymbol{k},\lambda}\sqrt{\frac{\omega_k}{2}}\boldsymbol{e}_\lambda(\boldsymbol{k})\left(a_{\boldsymbol{k},\lambda}e^{-i\omega_k t} - a^\dagger_{-\boldsymbol{k},\lambda}e^{i\omega_k t}\right)e^{i\boldsymbol{k}\cdot\boldsymbol{x}},\\
\boldsymbol{B}(t,\boldsymbol{x}) =& i\sqrt{\frac{\hbar}{\epsilon_0 L^3 c^2}}\sum_{\boldsymbol{k},\lambda}\sqrt{\frac{\omega_k}{2}}\boldsymbol{\kappa}\times\boldsymbol{e}_\lambda(\boldsymbol{k})\left(a_{\boldsymbol{k},\lambda}e^{-i\omega_k t} + a^\dagger_{-\boldsymbol{k},\lambda}e^{i\omega_k t}\right)e^{i\boldsymbol{k}\cdot\boldsymbol{x}},
\end{aligned} \tag{2}$$

where $\boldsymbol{\kappa} = \boldsymbol{k}/\|\boldsymbol{k}\|$, $\omega_k = c\|\boldsymbol{k}\|$ and $\boldsymbol{k} \in \frac{2\pi}{L}\mathbb{Z}^3$. The polarization vectors $\boldsymbol{e}_\lambda(\boldsymbol{k})$ form an orthonormal basis of the the two-dimensional space orthogonal to $\boldsymbol{k}$. In other words $\boldsymbol{e}_\lambda(\boldsymbol{k})\cdot\boldsymbol{k} = 0$ and $\boldsymbol{e}_\lambda(\boldsymbol{k})\cdot\boldsymbol{e}_{\lambda'}(\boldsymbol{k}) = \delta_{\lambda,\lambda'}$. Moreover, we assume that $\boldsymbol{e}_\lambda(-\boldsymbol{k}) = \boldsymbol{e}_\lambda(\boldsymbol{k})$.

a) *As a start, show that*

$$\int \|\boldsymbol{E}(t,\boldsymbol{x})\|^2 d^3x = \frac{\hbar}{\epsilon_0}\sum_{\boldsymbol{k},\lambda}\frac{\omega_k}{2}\left(a_{\boldsymbol{k},\lambda}a^\dagger_{\boldsymbol{k},\lambda} - a_{\boldsymbol{k},\lambda}a_{-\boldsymbol{k},\lambda}e^{-2i\omega_k t} - a^\dagger_{-\boldsymbol{k},\lambda}a^\dagger_{\boldsymbol{k},\lambda}e^{2i\omega_k t} + a^\dagger_{-\boldsymbol{k},\lambda}a_{-\boldsymbol{k},\lambda}\right). \tag{3}$$

Hint: Again, this is the classical setting, so $a_{\boldsymbol{k},\lambda}$ are complex number, not operators. Consequently, the $\dagger$ in $a^\dagger_{-\boldsymbol{k},\lambda}$ simply means complex conjugation. Note that $\int e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}}d^3x = L^3\delta_{\boldsymbol{k},\boldsymbol{k}'}$, since we integrate with respect to a box with side-length L.

(4 points)

b) *Show that* (1) *can be rewritten as*

$$H_{\text{em}} = \sum_{\boldsymbol{k},\lambda}\hbar\omega_k|a_{\boldsymbol{k},\lambda}|^2.$$

Hint: Do a similar thing as in a) but for the $\int\|\boldsymbol{B}(t,\boldsymbol{x})\|^2 d3x$ term. Recall the general identity $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})$.

(4 points)

Remark: Recall that the energy of a photon with angular frequency ω_k is $\hbar\omega_k$.

c) In a similar fashion as we in b) rewrote energy of the field in terms of $a_{\boldsymbol{k},\lambda}$, we can also rewrite the momentum carried by the field. *Show that*

$$\epsilon_0\int \boldsymbol{E}(t,\boldsymbol{x})\times\boldsymbol{B}(t,\boldsymbol{x})d^3x = \sum_{\boldsymbol{k},\lambda}\hbar\boldsymbol{k}|a_{\boldsymbol{k},\lambda}|^2.$$

Hint: Recall the general identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$. When you reach something looking somewhat similar to (3), think of which terms are symmetric or anti-symmetric when we swap $\boldsymbol{k}$ to $-\boldsymbol{k}$. What happens when you sum over such terms?

(4 points)

Remark: Recall that the momentum of a photon with wave-vector $\boldsymbol{k}$ is $\hbar \boldsymbol{k}$.

2 Taming an infinity

In this exercise we turn to the quantized electromagnetic field, and consider one of the infinities that emerges in this quantization. In the lecture (see section 3.3.1 in the lecture notes) we learn that the expected electric field, $\langle \boldsymbol{E}(\boldsymbol{x}) \rangle$, at any given point $\boldsymbol{x}$ is zero for any Fock state with respect to the momentum modes (and thus also for the vacuum state), but that the strength of the fluctuations in the field, $\langle \|\boldsymbol{E}(\boldsymbol{x})\|^2 \rangle$, diverges to infinity for the vacuum state. In order to avoid that you loose sleep from fear that you suddenly might explode due to some large vacuum fluctuations, we will here show that this infinity can be tamed. One can think of $\langle \boldsymbol{E}(\boldsymbol{x}) \rangle$ and $\langle \|\boldsymbol{E}(\boldsymbol{x})\|^2 \rangle$ as the effects on a point-charge at location $\boldsymbol{x}$. However, let us now instead consider the force

$$\boldsymbol{F} = \int \rho(\boldsymbol{x}) \boldsymbol{E}(\boldsymbol{x}) d^3 x,$$

on some charge distribution $\rho(\boldsymbol{x})$, where the electric field operator is

$$\boldsymbol{E}(\boldsymbol{x}) = i \sqrt{\frac{\hbar}{\epsilon_0 L^3}} \sum_{\boldsymbol{k},\lambda} \sqrt{\frac{\omega_k}{2}} \boldsymbol{e}_\lambda(\boldsymbol{k}) \left(a_{\boldsymbol{k},\lambda} - a^\dagger_{-\boldsymbol{k},\lambda} \right) e^{i \boldsymbol{k} \cdot \boldsymbol{x}},$$

which is the same operator as in (2), but where we have put $t = 0$. (Hence, we consider the electric field at one single moment.)

a) In the lecture we show that expectation value of $\boldsymbol{E}(\boldsymbol{x})$ is zero for any Fock-state $|n_1, n_2, \ldots\rangle$ for all the $(\boldsymbol{k}, \lambda)$-modes. Here we wish to show the same for the force $\boldsymbol{F}$. *Hence, show that* $\langle n_1, n_2, \ldots | \boldsymbol{F} | n_1, n_2, \ldots \rangle = 0$ *for any Fock-state* $|n_1, n_2, \ldots\rangle$. **(3 points)**

Remark: As a special case, we thus get that the expected force is zero for the vacuum state.

b) Now we turn to the expectation value of $\|\boldsymbol{F}\|^2$ with respect to the vacuum state $|\mathrm{vac}\rangle$. This measures the magnitude of the fluctuations in the vacuum field. *Show that*

$$\langle \mathrm{vac} | \, \|\boldsymbol{F}\|^2 \, | \mathrm{vac} \rangle = \frac{\hbar}{2 \epsilon_0 L^3} \sum_{\boldsymbol{k},\lambda} \omega_k |\tilde{\rho}(\boldsymbol{k})|^2, \tag{4}$$

where $\tilde{\rho}(\boldsymbol{k}) = \int \rho(\boldsymbol{x}) e^{i \boldsymbol{k} \cdot \boldsymbol{x}} d^3 x$.

(5 points)

Remark: This result means that if $|\tilde{\rho}(\boldsymbol{k})|^2$ goes to zero sufficiently fast as $\|\boldsymbol{k}\| \to \infty$, then the sum (4) becomes finite. Hence, the infinity can be tamed. Note also that the more smeared out $\rho(\boldsymbol{x})$ is, the faster the Fourier transform $\tilde{\rho}(\boldsymbol{k})$ decays, and thus the smaller the effect of the fluctuations.