Classical Mechanics

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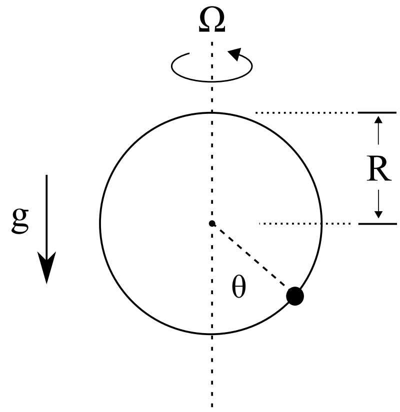
Exercise sheet 10 **Due: Thursday January 21 at 24:00**

1 Bead on a rotating hoop

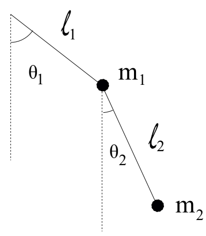


This exercise illustrates that the Lagrangian method also works for systems where the constraints depend on time. We consider a bead of mass m, which slides without friction on a hoop of radius R that rotates around its vertical axis with a constant angular speed Ω. Hence, the bead is constrained to only move on the hoop, but the hoop does in turn rotate. The bead is affected by gravity.

a) *Derive the Lagrangian expressed in terms of the angle θ that the bead makes with the vertical downwards direction.* **(3 points)**

b) *Find the equation of motion of the bead.* **(2 points)**

2 Double pendulum without gravity



Consider a double-pendulum where the two masses move in a plane. Mass m_1 is attached to a fixed point via a massless rod of length ℓ_1. A second mass m_2 is in turn attached by a massless rod of length ℓ_2 to mass m_1. In this problem we assume that the masses are *not* affected by gravity.

a) *Show that the Lagrangian with respect to the angles θ_1 and θ_2 as in the figure can be written as*

$$L = A\dot{\theta}_1^2 + B\dot{\theta}_2^2 + C\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1),$$

and determine the constants A, B and C. Note that the since there is no gravity, there is no potential energy. **(3 points)**

b) Recall from the lecture that a transformation $\vec{\Phi}^{(s)} : \mathbb{R}^n \to \mathbb{R}^n$ with $\vec{\Phi}^{(0)}(\vec{q}) = \vec{q}$ is a symmetry transformation of a Lagrangian $L(\vec{q}, \dot{\vec{q}})$ if

$$\frac{d}{ds}\Big|_{s=0} L\Big(\vec{\Phi}^{(s)}(\vec{q}(t)), \frac{d}{dt}\vec{\Phi}^{(s)}(\vec{q}(t))\Big) = 0.$$

Show that the transformation

$$(\theta_1, \theta_2) \mapsto \vec{\Phi}^{(s)}(\theta_1, \theta_2) = (\theta_1 + s, \theta_2 + s).$$

is a symmetry of the Lagrangian in a). **(2 points)**

c) *Derive the conserved quantity from Noether's theorem, with respect to the transformations $\vec{\Phi}^{(s)}$ in b).* **(3 points)**

3 Generalized Noether

The Noether-theorem that we employed in the previous exercise can be generalized. Recall from the lecture that a family of trajectories $\vec{q}^{(s)}(t)$ with $\vec{q}^{(0)}(t) = \vec{q}(t)$ satisfies the conditions for the generalized Noether theorem if there exists a function $\vec{\Delta}$ and a function f such that

$$\begin{aligned} \frac{d}{ds}|_{s=0}\vec{q}^{(s)}(t) &= \vec{\Delta}(\vec{q}(t), \dot{\vec{q}}(t), t) \qquad (1) \\ \frac{d}{ds}\Big|_{s=0} L(\vec{q}^{(s)}(t), \dot{\vec{q}}^{(s)}(t), t) &= \frac{d}{dt} f(\vec{q}(t), \dot{\vec{q}}(t), t). \end{aligned}$$

Suppose that we have a particle of mass m that lives on a line ($\mathbb{R}$) with coordinate q. This particle is affected by a time-dependent potential $V(q,t) = g(t)q$, for some (nice) function g.

a) *Determine the Lagrangian for the particle with respect to the coordinate q.* **(1 point)**

b) Consider the family of trajectories $q^{(s)}(t) = q(t) + s$. *Find functions Δ and f that satisfy the conditions of the generalized Noether-theorem in (1).* **(3 points)**

c) *Determine the conserved quantity corresponding to the transformation in b).* **(2 points)**

d) As an alternative to the generalized Noether-theorem, one can show that the quantity in c) is conserved by directly using the equation of motion of the particle. *Do that derivation.* **(1 point)**