Classical Mechanics

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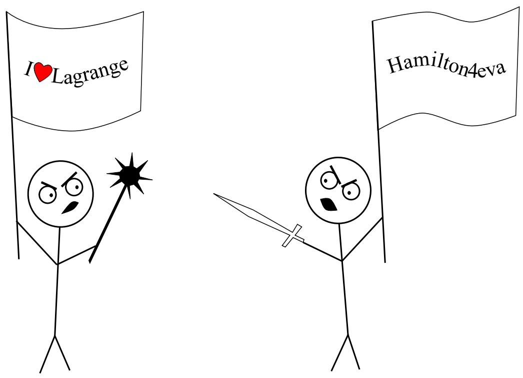
Exercise sheet 11 **Due: Thursday January 28 at 24:00**



1 From Lagrange to Hamilton

For the following Lagrangians, derive the Hamilton functions, and the Hamilton equations.

a)

$$L(\theta, \dot{\theta}) = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}R^2\Omega^2\sin^2\theta + mgR\cos\theta$$

(2 points)

Comment: This Lagrangian comes from an earlier exercise. Can you see which one?

b)

$$L(r, \theta, \dot{r}, \dot{\theta}) = \frac{m}{2}(1 + \frac{\alpha^2}{r^6})\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 + \frac{mg\alpha}{2r^2}$$

(3 points)

2 Particle in an electromagnetic field

For a particle with mass m, charge e, and position $\vec{r}$, which moves in an electromagnetic field, the Lagrangian can be written as

$$L(\vec{r}, \dot{\vec{r}}) = \frac{m}{2}\dot{\vec{r}}^2 - e\phi(\vec{r}, t) + e\vec{A}(\vec{r}, t) \cdot \dot{\vec{r}}, \tag{1}$$

where $\phi(\vec{r}, t)$ is a real-valued function (the scalar potential), and $\vec{A}(\vec{r}, t)$ is a vector-valued function (the vector-potential)[1].

Please do not panic if you are not familiar with electromagnetism and vector-potentials, think of (1) as simply being yet another (maybe strange looking) Lagrangian of a particle.

a) Introduce the Cartesian coordinates $\vec{r} = (r_1, r_2, r_3)$ and $\vec{A} = (A_1, A_2, A_3)$. *Show that the Euler-Lagrange equations obtained from the Lagrangian* (1) *can be written as*[2]

$$m\ddot{r}_j + e\left(\frac{\partial\phi}{\partial r_j} + \frac{\partial A_j}{\partial t}\right) - e\sum_{k=1}^{3}\dot{r}_k\left(\frac{\partial A_k}{\partial r_j} - \frac{\partial A_j}{\partial r_k}\right) = 0, \quad j = 1, 2, 3. \tag{2}$$

[1] The electric field can be obtained as $\vec{E}(\vec{r}, t) = -\nabla\phi - \frac{\partial}{\partial t}\vec{A}$ and the magnetic field as $\vec{B}(\vec{r}, t) = \nabla \times \vec{A}$.

[2] The more standard way of writing (2) is $m\ddot{\vec{r}} = e\vec{E} + e\dot{\vec{r}} \times \vec{B}$.

Hint: It can be useful to first rewrite (1) in terms of the Cartesian components

$$L(\vec{r},\dot{\vec{r}}) = \frac{m}{2}\sum_{k=1}^{3}\dot{r}_k^2 - e\phi(\vec{r},t) + e\sum_{k=1}^{3}A_k(\vec{r},t)\dot{r}_k.$$

(2 points)

b) Let $\chi(\vec{r},t)$ be a real-valued function (a scalar function), and suppose that we change ϕ and $\vec{A}$ into the new functions ϕ' and $\vec{A}'$ by

$$\phi' = \phi - \frac{\partial\chi}{\partial t}, \quad \vec{A}' = \vec{A} + \nabla\chi.$$

Let L' be the new Lagrangian that is obtained if we substitute ϕ and $\vec{A}$ in (1) by ϕ' and $\vec{A}'$. *Show that L and L' only differ by a total time-derivative of some function of $\vec{r}$ and t.* **(2 points)**

Comment: This type of mapping of the potentials is called a gauge-transformation. Recall that the addition of a total time-derivative to the Lagrangian does not change the Euler-Lagrange equations (and thus does not change the evolution of the system).

c) *Derive the Hamilton function.* **(2 points)**

d) *Derive the Hamilton equations.* **(2 points)**

Comment: This is a non-trival example of the application of the Lagrange and Hamilton formalism, and also gives a first glimpse of the notion of gauge invariance, which you will encounter in other courses.

3 The algebra of Poisson brackets

For two functions $f(q_1,\ldots,q_N,p_1,\ldots,p_N,t)$ and $g(q_1,\ldots,q_N,p_1,\ldots,p_N,t)$, the Poisson bracket is defined as[3]

$$\{f,g\} = \sum_{n=1}^{N}\left(\frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n} - \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n}\right). \tag{3}$$

In the lecture we discussed five properties that the Poisson brackets satisfies (see video 4.2). When calculating with Poisson brackets it is often enough to make use of these relations, without having to think about the definition (3), as we shall see in this exercise.

a) *Show that*

$$\{q_j, p_k^n\} = -np_k^{n-1}\delta_{jk}, \quad \{p_j, q_k^n\} = nq_k^{n-1}\delta_{jk}, \quad n = 1,2,3,\ldots$$

Hint: This is efficiently shown via an induction proof. **(3 points)**

b) The angular momentum of a particle with position $\vec{q}$ and momentum $\vec{p}$, is given by $\vec{L} = \vec{q}\times\vec{p}$. For $\vec{L} = (L_1, L_2, L_3)$, one can write $L_j = \sum_{kl}\epsilon_{jkl}q_k p_l$, where ϵ_{jkl} is the Levi-Civita symbol[4].

[3] The definition of the Poisson bracket occurs in two variations that differ in the choice of the overall sign. This choice affects the sign of the bracket $\{p_k, q_l\}$.

[4] The Levi-Civita symbol in three dimensions is $\epsilon_{jkl} = \begin{cases} 1 & (jkl) = (123),(312),(231) \\ -1 & (jkl) = (213),(321),(132) \\ 0 & \text{else} \end{cases}.$

Show that

$$\{q_n, L_j\} = -\sum_k \epsilon_{njk} q_k, \quad \{p_n, L_j\} = -\sum_l \epsilon_{njl} p_l, \quad j, n = 1, 2, 3,$$

and

$$\{\vec{q}^2, L_j\} = 0, \quad \{\vec{p}^2, L_j\} = 0, \quad j = 1, 2, 3.$$

Hint: Note that $(\vec{a} \times \vec{b})_j = \sum_{kl} \epsilon_{jkl} a_k b_l$ and $\vec{a} \times \vec{a} = 0$. Note also that whenever you permute two indices in ϵ_{jkl}, then it changes sign, e.g., $\epsilon_{jkl} = -\epsilon_{kjl} = \epsilon_{klj}$.

(4 points)