Classical Mechanics

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Exercise sheet 12 **Due: Thursday February 4 at 24:00**

1 Conserved quantities via Poisson brackets

This exercise exemplifies that Poisson brackets can be used in order to identify conserved quantities.

Suppose that we have two particles of mass m that move in three-dimensional space ($\mathbb{R}^3$), and interact with each other via a quadratic potential. This can be described with the following Hamilton function:

$$H(\vec{q}_1, \vec{q}_2, \vec{p}_1, \vec{p}_2) = \frac{1}{2m}\vec{p}_1^2 + \frac{1}{2m}\vec{p}_2^2 - \alpha\|\vec{q}_1 - \vec{q}_2\|^2.$$

Let $\vec{L}^{(1)} = \vec{q}_1 \times \vec{p}_1$ and $\vec{L}^{(2)} = \vec{q}_2 \times \vec{p}_2$ be the angular momentum vectors of particle 1 and 2, respectively.

a) *Show that*

$$\left\{H, L_j^{(1)} + L_j^{(2)}\right\} = 0, \quad j = 1, 2, 3. \tag{1}$$

Hint: There are various observations that make the derivation quicker. For example, $\{f(\vec{q}_1, \vec{p}_1) + f(\vec{q}_2, \vec{p}_2), g(\vec{q}_1, \vec{p}_1) + g(\vec{q}_2, \vec{p}_2)\} = \{f(\vec{q}_1, \vec{p}_1), g(\vec{q}_1, \vec{p}_1)\} + \{f(\vec{q}_2, \vec{p}_2), g(\vec{q}_2, \vec{p}_2)\}$. The relations that you proved in exercise 3b on sheet 11 could be useful. Recall that the cross-product can be written $(\vec{a} \times \vec{b})_j = \sum_{kl} \epsilon_{jkl} a_k b_l$, where ϵ_{jkl} is the Levi-Civita symbol.

(4 points)

b) *In more physical terms, what is it that you have proved with equation (1)?* **(1 point)**

2 Solving the equations of motion via canonical transformations

In the lecture we used a canonical transformation in order to solve the equations of motion of the harmonic oscillator. Here we shall use this technique on a more complicated Hamiltonian, namely

$$H(q, p) = \frac{1}{2}p^2 q^4 + \frac{1}{2q^2}. \tag{2}$$

The strategy will be to find a canonical transformation that turns (2) into a harmonic oscillator. By solving the latter, we can then work backwards in order to find an explicit formula for the evolution caused by (2). The question is how to find that first canonical transformation. Here we shall create a family of canonical transformations, and hope that it is big enough to achieve what want.[1]

a) Since we want to turn some combinations of powers of q and p (in (2)) into some other powers of Q and P of the harmonic oscillator (in (4)), one idea would be to try some transformations that combine powers of q and p. The problem is that arbitrary expressions would generally not result in canonical transformations, so we have to determine which combinations of powers yield valid canonical transformations.

Consider the family of transformations from (q, p) to (Q, P) defined by[2]

$$P = \alpha p^\beta q^\gamma, \quad Q = q^\delta, \tag{3}$$

[1] There is no guarantee that this works. Basically, it is like shooting from the hip; if we are lucky we hit.

[2] As I said, this is a wild guess. One could of course also take larger families of transformations, with more free parameters, but let's not get too extreme.

where α, β, γ and δ are constants. *What are the conditions on α, β, γ and δ for (3) to be a canonical transformation?*

Hint: Recall the characterization of canonical transformations in terms of Poisson brackets, and the definition of the Poisson bracket.

(3 points)

b) *Use the result from a) to find a canonical transformation from (q, p) to (Q, P) that transforms the Hamiltonian in (2) into the Hamilton function of the Harmonic oscillator*

$$H(Q, P) = \frac{1}{2}P^2 + \frac{1}{2}Q^2. \tag{4}$$

(3 points)

c) *Use the result in b) in order to solve the equations of motion generated by (2).*

(3 points)

3 Generating functions for canonical transformations

Generating functions can be used to find canonical transformations from coordinates $(\vec{q}, \vec{p})$ to a new set of coordinates $(\vec{Q}, \vec{P})$. In the lecture we discussed generating functions of the form $F(\vec{q}, \vec{Q})$. Here we are going to apply this machinery. Consider such a function $F(q, Q)$ of the old coordinate q and the new coordinate Q. This function does implicitly define a canonical transformation between (q, p) and (Q, P) via the two equations

$$p = \frac{\partial F}{\partial q}, \quad P = -\frac{\partial F}{\partial Q}. \tag{5}$$

a) Consider the function

$$F(q, Q) = \frac{m\omega}{2} q^2 \frac{1}{\tan Q},$$

where m and ω are some constants. *Use the relations (5) in order to express q and p as functions of Q and P.*

Hint: The relations (5) give p and P as functions of q and Q, and you have to transform these so that you obtain q and p as functions of Q and P. Do not worry about whether the square roots are well defined, or about the sign of the roots. **(3 points)**

b) Consider the Hamilton function for the harmonic oscillator

$$H(q, p) = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} q^2.$$

Express H in terms of the new variables Q and P. What is the solution of the corresponding equations of motion of Q and P? Transform the solution back to the original (q, p) and thus obtain the solutions to the equations of motion of the harmonic oscillator. **(3 points)**