Classical Mechanics

David Gross, David Wierichs, Markus Heinrich, Johan Åberg

Exercise sheet 2 **Due: Thursday November 12 at 24:00**

1 Vector fields in phase space from conservative forces

Recall from the lecture that Newton's equations of motion

$$\ddot{r} = \frac{1}{m} F\big(r(t), \dot{r}(t)\big)$$

can be rewritten as two coupled first order differential equations

$$\frac{d}{dt}\vec{x} = \vec{\mathcal{F}}(\vec{x}), \quad \vec{\mathcal{F}} = \begin{pmatrix} \mathcal{F}_r \\ \mathcal{F}_v \end{pmatrix}, \quad \vec{x}(t) = \begin{pmatrix} r(t) \\ v(t) \end{pmatrix}, \tag{1}$$

where $v(t) = \dot{r}(t)$. Here $\vec{\mathcal{F}}$ is the vector field in phase space associated to the force F.

Recall that the divergence $\nabla \cdot \vec{\mathcal{F}} = \frac{\partial \mathcal{F}_r}{\partial r} + \frac{\partial \mathcal{F}_v}{\partial v}$ at $\vec{x}$ in some sense measures to what extent there is a source of the field $\vec{\mathcal{F}}$ at $\vec{x}$. Alternatively, one can say that if the flow 'expands', then the divergence is positive, while if it 'contracts', then the divergence is negative.

In the following we assume that the force F only depends on the position r (but not on v or t). For one-dimensional systems, this is an example of what one often refers to as a conservative force. (Beware that in higher dimensions a force is not necessarily conservative just because it only depends on the position.)

a) *Suppose that the force F only is a function of r (but not of $\dot{r}$ or t). What is the divergence of the corresponding field $\vec{\mathcal{F}}$? What can you conclude about the flow in phase space resulting from such a force-function?* **(2 points)**

b) Suppose that the force is given by $F(r) = \gamma r$ for some constant $\gamma > 0$. *Determine the vector field $\vec{\mathcal{F}}$ corresponding to this force.* **(2 points)**

Remark: The system corresponding to this particular force is sometimes referred to as the inverted oscillator. (Can you guess why? Compare with the force-function for the harmonic oscillator.)

c) *Make a qualitative sketch of the vector field $\vec{\mathcal{F}}$ in b). For the sake of simplicity, put $\gamma/m = 1$.*

Hint: Consider the field along the lines $s(1,1), s(1,-1), s(1,0), s(0,1)$ for $s \in \mathbb{R}$. **(2 points)**

d) In the lecture you discussed how one can use numerical methods to obtain the solutions to the evolution. However, in this case we can solve it directly. *For the same force as in b), and with $\gamma/m = 1$, solve the differential equation (1). In the special cases that the initial point lies on the diagonals $s(1,1)$ or $s(1,-1)$, compare with the result in c).*

Hint: In the current case we can rewrite (1) as $\frac{d}{dt}\left[\begin{smallmatrix} r \\ v \end{smallmatrix}\right] = \boldsymbol{M}\left[\begin{smallmatrix} r \\ v \end{smallmatrix}\right]$, where $\boldsymbol{M}$ is a 2×2 matrix. Recall that we can solve linear differential equations by matrix exponentiation, such that the solution is $\left[\begin{smallmatrix} r(t) \\ v(t) \end{smallmatrix}\right] = \exp(t\boldsymbol{M})\left[\begin{smallmatrix} r(0) \\ v(0) \end{smallmatrix}\right]$. There are different methods to evaluate the matrix exponential, but in the current case it could be favorable to use a Taylor expansion of the exponential. Due to the particular form of $\boldsymbol{M}$, you will find something on the form $\exp(t\boldsymbol{M}) = f_1(t)\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] + f_2(t)\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, for some functions f_1 and f_2. **(3 points)**

Comment: The purpose of this exercise, and the next, is to illustrate the notion of vector fields that was introduced in the lecture in order to describe the dynamics in phase space.

2 Vector fields in phase space for non-conservative forces

If the force does depend non-trivially on $\dot{r}$ we would have a non-conservative force, where a typical example would be friction.

a) Suppose that the force is given by

$$F(r, \dot{r}) = -\alpha \dot{r}, \tag{2}$$

where α is a constant. If $\alpha > 0$ one can see that the force $-\alpha\dot{r}$ is always opposite to the direction of motion, and thus will tend to slow the particle down. In other words, if $\alpha > 0$ there is friction. If $\alpha < 0$, then we have some form of anti-friction, where the particle speeds up in it's direction of motion.

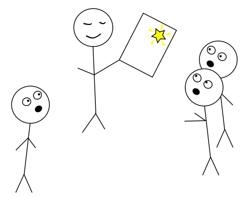
Determine the vector field $\vec{\mathcal{F}}$ corresponding to the force in (2). **(2 points)**

b) *Draw a rough sketch of the vector field for the case $\alpha > 0$. Analogously draw a sketch for the case $\alpha < 0$. For the sake of simplicity, put $|\alpha|/m = 1$.* **(3 points)**

c) *Determine the divergence of the vector-field $\vec{\mathcal{F}}$ resulting from (2). How does the sign of the divergence depend on the sign of α? What can you thus say about the contraction or expansion of the flow in phase space for the two cases?* **(3 points)**

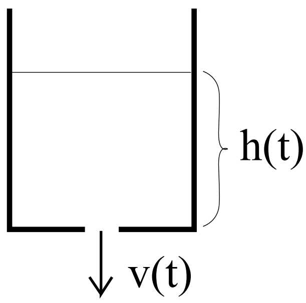
d) This is again an example of a system where we can find explicit solutions. *For the initial position $r(0)$ and velocity $v(0)$, find the solution $r(t), v(t)$. Put $|\alpha|/m = 1$. Compare with the sketches of the field in b).* **(3 points)**

3 Gold star exercise: Non-unique solutions can be physically relevant



This exercise gives absolutely no points! However, by solving it you do of course become the center of admiration and envy for all your peers. Moreover, your family will (obviously) bask in unbound glory for at least three generations. We might also discuss Gold star exercises on the Thursday online-sessions.

In the lecture it was mentioned that Picard-Lindelöf's theorem guarantees that the solutions of differential equations are unique. At first sight, one might think that non-uniqueness is un-physical and implies that something is wrong with the theory. However, there are situations where the non-uniqueness of solutions makes sense physically. In this exercise we are going to explore such a scenario.



Imagine a bucket with a hole in the bottom. You are told that the bucket initially was filled with water. Suppose that you get to see the empty bucket with a puddle of water under it. The question is, can you tell when the bucket was full? Intuitively, the answer is no, since the bucket could get empty at any time. Hence, once empty, we can no longer tell when it was full.

In the following we will construct a concrete (although somewhat crude) model of the bucket. Let $h(t)$ be the height of the water in the bucket (counted from the bottom). Let $v(t)$ be the velocity of the water flowing out of the hole. We assume that the water is incompressible. Let a be the area of he hole and A the cross-sectional area of the bucket.

a) *Argue that*

$$av(t) = -A\frac{d}{dt}h(t). \tag{3}$$

(0 points)

b) Consider that the water in the bucket decreases with the mass ΔM. This effectively removes this mass from the top of the water column at height h, and thus corresponds to the loss of potential energy ΔMgh. Assuming that no energy is lost by friction, this could be put equal to the kinetic energy $\Delta Mv^2/2$ of the mass leaving the bucket[1]. By equating these we would get

$$v^2 = 2gh. \tag{4}$$

Combine (3) *and* (4) *to obtain a differential equation for* $h(t)$. **(0 points)**

c) *Given that the height initially is* h_0*, determine the time that it takes until the bucket is empty.* **(0 points)**

d) *Compare with the solutions to the model of friction in Exercise 2d). Given that the velocity initially is non-zero, how long time does it take before the velocity is zero? Can you see the crucial difference between these two systems, concerning uniqueness or non-uniqueness of the solutions?* **(0 points)**

[1] Alternatively, one can argue via Bernoulli's equation, which describes the relation between the velocity, potential energy, and pressure along a streamline of a stationary, incompressible, laminar and non-viscous flow.