Classical Mechanics

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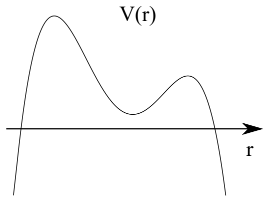
Exercise sheet 3 **Due: Thursday November 19 at 24:00**

1 Phase space portrait

Consider a particle of mass m that moves in one dimension under the influence of a potential $V(r)$ of the form as in the figure below.



In this exercise we are going to explore the phase space portrait of this potential. Since we only are given a qualitative description of the potential V, you can of course only give a qualitative sketch of the portrait.

a) Before turning to phase space, let us first consider the potential. Recall that an equilibrium point is such that if we put the particle there with zero velocity, then it will stay there indefinitely. *What are the equilibrium points of V, and which are stable and which are unstable?*

(2 points)

b) Let us now turn to the main task, namely to give a qualitative picture of the phase space portrait of the potential V. *Draw a sketch of the portrait, which includes the following:*

- *All fixpoints. Indicate which fixpoints are elliptic and which are hyperbolic.*
- *All separatrices.*
- *For each of the "regions" separated by the separatrices, indicate the direction of the flow.*

Hints: In general, when drawing a phase space portrait, it can be a good idea to first find the fixpoints, and then identify whether they are elliptic or hyperbolic. For each isolated hyperbolic fixpoint there are two ingoing and two outgoing separatrices (although one that goes out can turn around and go in again). Each elliptic fixpoint is enclosed by loops corresponding to bound solutions. To assign the direction of the flow, think of which direction of motion that the upper half of the phase plane corresponds to.
For the particular potential that we consider here, keep in mind that the two maxima of the potential are at different energies. **(4 points)**

2 A complicated way to solve the harmonic oscillator

In the lecture we discussed a general method to determine the evolution of one-dimensional systems. Here we will apply this method on the harmonic oscillator[1]. In the lecture we showed that

$$t(r) = \sqrt{\frac{m}{2}} \int_{r_0}^{r} \frac{1}{\sqrt{E - U(r')}} dr', \tag{1}$$

[1]For the harmonic oscillator there are much simpler methods to find the solutions, but for the sake of illustration we use this method anyway.

where r_0 is the initial position of the system. In principle one can then obtain the solution $r(t)$ by inverting the function $t(r)$. We consider the harmonic oscillator $U(r) = \frac{1}{2}kr^2$ for $k > 0$, and we put the initial position to $r_0 = 0$.

a) *Evaluate the integral in (1) for the harmonic oscillator.*

Hint: Use a change of variables $r'' = r'\sqrt{\frac{k}{2E}}$ and find the primitive function (anti-derivative), and the explicit function $t(r)$. **(3 points)**

b) *By inverting the function $t(r)$, find an expression of the form $r(t) = A\sin(Bt)$, and determine the constants A and B.* **(2 points)**

3 Period of the mathematical pendulum: First correction for small oscillations

When we discussed the mathematical pendulum in the lecture, we found that the solution in terms of the time $t(\phi)$ that it takes to reach angle ϕ, starting from angle 0 is

$$t(\phi) = \sqrt{\frac{ml^2}{2}} \int_0^{\phi} \frac{1}{\sqrt{E - 2mgl\sin^2(\phi'/2)}} d\phi', \qquad E = 2mgl\sin^2(\phi_{\max}/2), \tag{2}$$

where $\phi_{\max}$ is the angle of the turning point of the oscillation. We wish to find an expression for the period T as a function of $\phi_{\max}$. More precisely, we wish to find the first two non-zero terms in a Taylor expansion of $T(\phi_{\max})$.

a) *Express the period T of the oscillations in terms of the function $t(\phi)$.* **(1 point)**

b) As the first step we will make a change of variables in (2) to the new variable θ, such that

$$\sin\theta = \frac{\sin(\phi/2)}{\sin(\phi_{\max}/2)}.$$

Show that the period can be written

$$T(\phi_{\max}) = A\int_B^C \frac{1}{\sqrt{1 - \sin^2\theta\sin^2(\phi_{\max}/2)}} d\theta, \tag{3}$$

and determine the constants A, B and C.

(4 points)

c) Next, we use the integral in (3) as the starting point for our approximations. *Show that*

$$T(\phi_{max}) = \alpha\left[1 + \beta\phi_{\max}^2\right] + O(\phi_{\max}^3),$$

and determine the constants α and β.

Hint: Apply the Taylor-expansion $\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + O(x^2)$, up to first order, to the integrand of (3), and evaluate the resulting integral. Next, use the fact that $\sin(\phi_{\max}/2) = \phi_{\max}/2 + O(\phi_{\max}^3)$.

(4 points)