Classical Mechanics

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Exercise sheet 4 Due: Thursday November 26 at 24:00

1 Barge on a river

In the lecture we discussed various notions in relation to conservative and non-conservative forces, and here we explore these concepts.

Imagine that you are operating a barge that is attached to huge ropes that stretch over a river. By pulling the ropes you can move the barge around. The flow of the river exerts a force on the barge which we will assume to be $\vec{F} = q\vec{v}$, where $\vec{v}$ is the velocity of the flow and $q > 0$ is some constant.

If the barge is pulled very slowly around on the river, we can assume that all of the force is directly transferred to the machinery that pulls the ropes. Suppose that the velocity profile of the current is $\vec{v} = v_0(1 - \frac{x^2}{a^2})\hat{y}$ for $-a \leq x \leq a$ and $v_0 > 0$. Here, $x = -a$ and $x = a$ are the two banks of the river and $\hat{y}$ denotes the unit vector in the positive y-direction. Hence, we have oriented the coordinate system such that the positive y-direction is in the direction of the flow of river, the x-axis is perpendicular to the river, and $x = 0$ is in the middle.

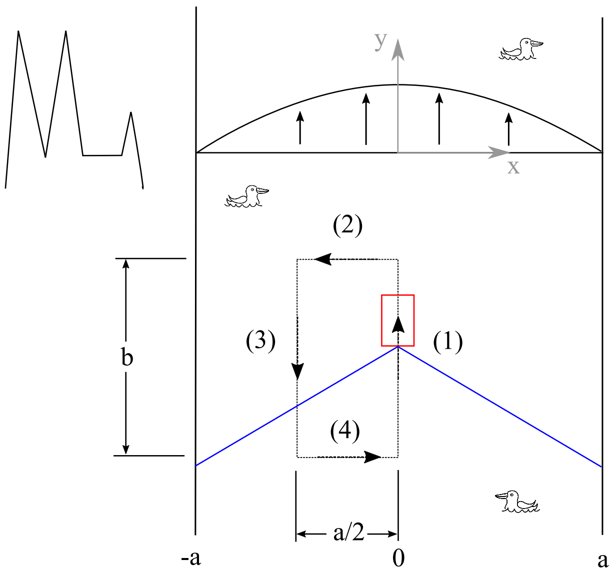

Figure 1: A barge (red) is attached to two huge ropes (blue) that stretch over a river and is moved along the indicated cycle mentioned in the text.

Next, let us assume that we want to move the barge along a closed cycle C as indicated in Fig. 1. The cycle consists of four straight lines described by the following sequence of points $(x, y) = (0, 0) \to (0, b) \to (-a/2, b) \to (-a/2, 0) \to (0, 0)$, where $a, b > 0$.

a) *Evaluate the curl $\vec{\nabla} \times \vec{F}$ in Cartesian coordinates. Is the force field conservative or non-conservative?* Think of the force field as being independent of z. **(2 points)**

b) *By evaluating the line integral of the force exerted by the river on the barge, determine how much energy that you would gain, or need to spend, in order to pull the barge once around the cycle C. Where does that energy come from?* **(3 points)**

c) *Confirm the result in (b) by using Stokes theorem.*

Hint: Recall that Stokes theorem says that $\oint_C \vec{F} \cdot d\vec{r} = \int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A}$, where the right hand side is the surface integral over an area S enclosed by the closed loop C on the left hand side[1]. Keep in mind that the orientations of the loop C and the surface S have to obey the right-hand rule. **(3 points)**

[1] I. e. $C = \partial S$

2 Maximum angular momentum for a circular orbit

As we have seen in the lecture, two-body problems (in $\mathbb{R}^3$) can be simplified quite dramatically, resulting in an effective one-dimensional motion of a fictitious particle. Consider a two-body problem, with reduced mass μ, where the interaction potential can be expressed in terms of the relative coordinate $\vec{r}$ as

$$U(r) = -U_0 e^{-\lambda^2 r^2}, \quad r = \|\vec{r}\|, \quad U_0 > 0, \quad \lambda > 0. \tag{1}$$

a) *Determine the effective potential U_{eff} corresponding to (1) for a given magnitude of the angular momentum $\|\vec{L}\| = \ell$.*

(1 point)

b) *Show that the radius r_0 of any circular orbit has to satisfy the equation*

$$\ell^2 = A r_0^B e^{-\lambda^2 r_0^2}, \tag{2}$$

and determine the constants $A \geq 0$ and $B \geq 0$. **(2 points)**

c) *Find the largest value of ℓ such that there exists a circular orbit.*

Hint: The right hand side of (2), regarded as a function of r_0, has one single maximum. **(2 points)**

3 The Laplace-Runge-Lenz vector

In the lecture we have discussed centrally symmetric potentials. In this exercise we are going to consider the special class of centrally symmetric potentials of the form

$$U(\vec{r}) = -\frac{k}{r}, \quad r = \|\vec{r}\|, \tag{3}$$

for some constant k, where examples are gravity and the Coulomb potential. The question of the dynamics of systems of this type is often referred to as the "Kepler problem".

In the lecture we demonstrated that the angular momentum is conserved for centrally symmetric potentials. Moreover, like for any other conservative force, the energy is conserved. Here we are going to show that for the Kepler problem, i.e., for potentials of the form (3), there exists an additional "accidental" conserved quantity, namely the Laplace-Runge-Lenz vector, which is defined as

$$\vec{A} = \mu \dot{\vec{r}} \times \vec{L} - \mu k \hat{r},$$

where μ is the reduced mass, $\hat{r}$ is the unit vector in the direction of $\vec{r}$, and $\vec{L}$ is the angular momentum. *Show that $\vec{A}$ is conserved, i.e., show that $\frac{d\vec{A}}{dt} = 0$.*

Hint: What is the force in the current case? How does the force relate to $\mu\ddot{\vec{r}}$? Keep in mind that $\vec{L}$ is conserved. Make use of the general relation for cross-products $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$. Show that $\vec{r} \cdot \frac{d\vec{r}}{dt} = r\frac{dr}{dt}$. **(7 points)**