Classical Mechanics

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Exercise sheet 5 **Due: Thursday December 3 at 24:00**

1 Scattering angle as a function of the impact parameter

In the lecture we discussed the scattering in the Kepler problem, i.e., scattering on a potential of the form

$$U(r) = -\frac{k}{r} \qquad r = \|\vec{r}\|.$$

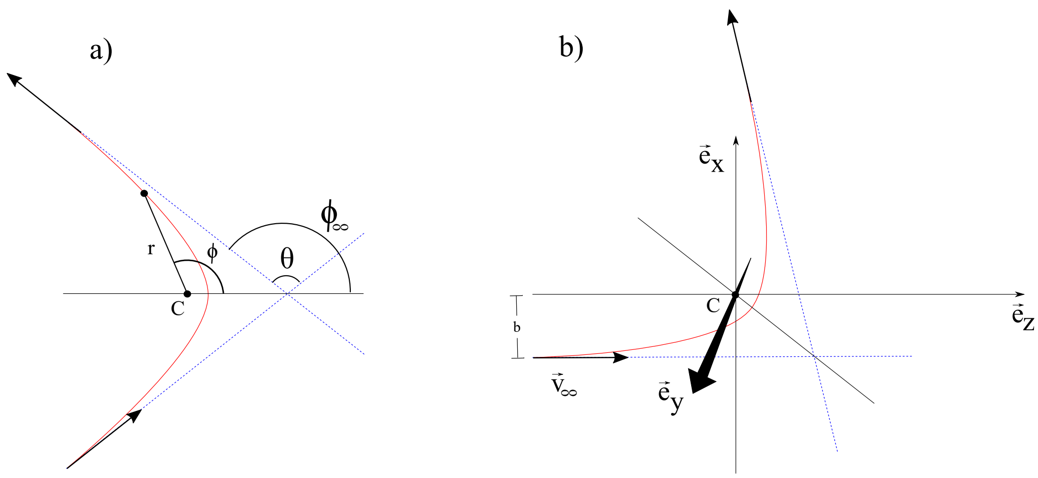
As you recall from the lecture, the solutions to the Kepler problem are

$$r = \frac{p}{1+\epsilon\cos\phi} \tag{1}$$

with r the distance from the origin (the focal point) and ϕ the angle as in figure a) below. In this exercise we are interested in scattering solutions, where the particle comes in from infinity and goes out to infinity again. More precisely, we are interested in the case $\epsilon > 1$, where the orbits form hyperbolas. The goal is to show that the scattering angle θ can be written as a function of the impact parameter b as

$$\theta = 2\arctan\left(\frac{k}{2Eb}\right). \tag{2}$$

For $\epsilon > 1$ there is an angle ϕ_∞ for which $r = +\infty$, indicated in figure a). In figure a) we also include the scattering angle θ. In figure b) we have introduced a coordinate system with origin at the focal point C, and basis elements $\vec{e}_x, \vec{e}_y, \vec{e}_z$. Here, $\vec{e}_z$ is aligned with the vector $\vec{v}_\infty$, which is the velocity of the incoming particle (at time $t = -\infty$) very far from C. The pair $\vec{e}_x, \vec{e}_z$ spans the plane of the motion of the particle (the ecliptic).



a) *Express the scattering angle θ in terms of the asymptotic angle ϕ_∞.*

Hint: Think about the angles in figure a). **(1 point)**

b) *Express the asymptotic angle ϕ_∞ in terms of ϵ*[1].

Hint: Think of equation (1).

(1 point)

[1]Take the smallest positive angle.

c) The angular momentum $\vec{L}$ is constant, and can be calculated along any point of the trajectory. We evaluate it in the limit $t = -\infty$. When the particle is at a position $\vec{r}$ very far from the scattering center C, we put as an approximation $\dot{\vec{r}} = \vec{v}_\infty$, and $\vec{r} \cdot \vec{e}_x = -b$. *Use this to express $\vec{L}$ in terms of the basis elements $\vec{e}_x, \vec{e}_y, \vec{e}_z$, as well as of b and $v_\infty = \|\vec{v}_\infty\|$.* **(2 points)**

d) From the lecture we know that we can write the eccentricity as

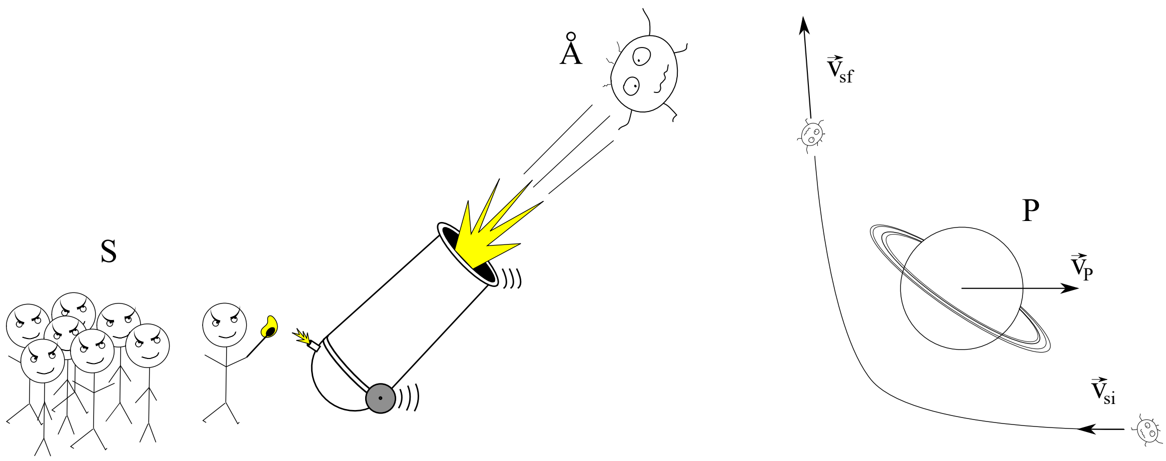
$$\epsilon = \frac{\|\vec{A}\|}{k\mu}, \quad \vec{A} = \mu\dot{\vec{r}} \times \vec{L} - \mu k\hat{r},$$

where $\vec{A}$ is the Runge-Lenz vector, and μ is the reduced mass. *Express ϵ in terms of the total energy E, the impact parameter b, and the constant k.*

Hint: Since $\vec{A}$ is conserved, one possible strategy to calculate $\|\vec{A}\|$ is to evaluate it in the limit $t = -\infty$, like we did for the angular momentum in c). In this limit, use the result in c), and think of what $\hat{r} = \vec{r}/\|\vec{r}\|$ approaches. What is the relation between the total energy E and v_∞ in this limit? **(3 points)**

e) *Combine the findings in a), b) and d) in order to derive* (2). **(3 points)**

2 Shooting something very far into space



A group of students S want to send an individual Å, who creates exercises in classical mechanics, very far into space. They use a cannon to shoot this object into space[2]. They have enough gunpowder to reach the necessary escape velocity from Earth. However, they want to make sure that Å disappears deeply into interstellar space, but at the same time they want to use as little gunpowder as possible, so that they have more money left for the party afterwards. Some of them have read a Wikipedia-article about a clever technique called the "gravitational slingshot´´ (or "gravity assist maneuver´´, or "swing-by´´). The idea is that one can increase the speed (relative to the sun) of a spacecraft (or an object Å) by letting it pass near a planet in a suitable way. The problem is that the group of students S are not quite sure how all of this works, so you have to help them to understand the general principles.

As mentioned above, the gravitational sling-shot works by letting object Å pass near a planet P. For the first step of the analysis we consider the system in the center of mass frame of Å and P[3]. We assume that the mass M of P is much larger than the mass m of Å. Hence, we can to a very

[2] Although they do intend to become dictators (as one can see on their eyebrows) they are still only doing dictator-internships, so they can unfortunately not afford a rocket.

[3] We ignore the influence from the sun.

good approximation put the center of mass at the center of P, and the relative mass μ equal to m. The total energy of Å is positive[4] since otherwise Å would be trapped in a periodic orbit around P[5]. Hence, Å follows a hyperbolic orbit ($\epsilon > 1$).

We let $\vec{v}_{pi}$ and $\vec{v}_{pf}$ denote the initial and final velocity of Å very far from P, *in the center of mass frame*. Note that the scattering angle θ (the same as in Exercise 1) is the angle by which the initial velocity $\vec{v}_{pi}$ is turned into the final velocity vector $\vec{v}_{pf}$. (Hence, we do for example have $\vec{v}_{pi} \cdot \vec{v}_{pf} = v_{pi} v_{pf} \cos\theta$.)

a) *Show that the initial speed $v_{pi} = \|\vec{v}_{pi}\|$ and final speed $v_{pf} = \|\vec{v}_{pf}\|$ are equal.* **(1 point)**

b) The goal is to determine the difference between the final kinetic energy $E_{\text{kin},f}$ of object Å and and the initial kinetic energy $E_{\text{kin},i}$, *with respect to the reference frame of the sun*. Let $\vec{v}_P$ denote the velocity of planet P (also in the frame of the sun). *Show that*

$$E_{\text{kin},f} - E_{\text{kin},i} = m(\vec{v}_{pf} - \vec{v}_{pi}) \cdot \vec{v}_P.$$

Hint: Let $\vec{v}_{si}$ and $\vec{v}_{sf}$ be the initial and final velocities of Å *in the reference frame of the sun*. What is the relation between $\vec{v}_{si}$ and $\vec{v}_{sf}$ and $\vec{v}_{pi}$ and $\vec{v}_{pf}$? **(2 points)**

c) To demonstrate the gravitational slingshot one can consider the special case that the initial velocity $\vec{v}_{si}$ of Å, in the frame of the sun, is anti-parallel to $\vec{v}_P$. In other words, Å initially moves in the opposite direction compared to planet P. Hence, we let $\vec{v}_{si} = -s\vec{v}_P$ for $s > 0$. *Show that*

$$E_{\text{kin},f} - E_{\text{kin},i} = m(1+s)\|\vec{v}_P\|^2\Big(1 - \cos\theta\Big),$$

where θ is the scattering angle that we discussed in d). What does this imply for the relation between $E_{\text{kin},f}$ and $E_{\text{kin},i}$? Where does the energy come from? **(3 points)**

[4] We follow the convention to put the potential energy to zero infinitely far away from P.

[5] Strictly speaking zero energy would also be allowed. Then, the orbit would be parabolic (and thus $\epsilon = 1$), but we skip that special case here.

3 Application of Kepler's laws

Kepler's first law states that the planets move along ellipses around the sun[6]. One way to describe the orbit is by using polar coordinates, where the radius r depends on the angle θ as

$$r(\phi) = \frac{p}{1 + \epsilon \cos \phi}, \tag{3}$$

where ϵ is the eccentricity of the ellipse, and p gives the size of the orbit. Kepler's second law says that the line joining the planet and the sun sweeps out equal areas during equal intervals of time. Kepler's third law states the length a of the semimajor axis of the ellipse relates to the period T of the orbit as

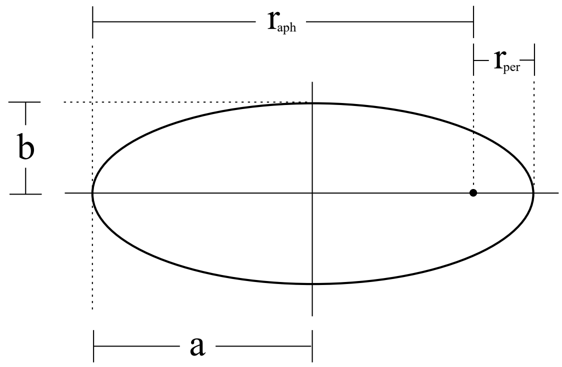


Figure 1: The semimajor axis a and the semiminor axis b of the ellipse. The shortest distance r_{per} to the sun (perihelion), and the longest distance r_{aph} (aphelion).

$$\frac{a^3}{T^2} = \frac{G(M+m)}{(2\pi)^2},$$

where M and m are the mass of the sun and the planet, respectively. Note that $G \approx 6.67 * 10^{-11} Nm^2/kg^2$.

a) Before landing, Apollo 11 was put in orbit around the moon. The mass of of Apollo 11 was 9979 kg and the period of the orbit was 119 min. The maximum and minimum distances from the center of the moon were 1861 km and 1838 km. *Use these data to estimate the mass of the moon.* **(2 points)**

b) Halley's comet moves in an elliptic orbit around the sun, with a period of 76 years. The eccentricity is $\epsilon = 0.97$. The mass of the sun is about $M = 2.0 * 10^{30} kg$. *Use these data to determine the distance from the sun at perihelion (when the comet is the closest to the sun) and aphelion (when it is the most far away).*

Hint: We can ignore the mass of the comet compared to the mass of the sun. **(2 points)**

Comment: The purpose of this exercise is to demonstrate some uses of Kepler's laws.

[6] Kepler's laws is not only applicable to things orbiting around the sun, but can also be applied also to other constellations. Strictly speaking, Eq. (3) describes the motion around the center of mass of the two bodies. However, in our case $M \gg m$ and thus the center of mass is approximately the position of the heavier body.