Classical Mechanics

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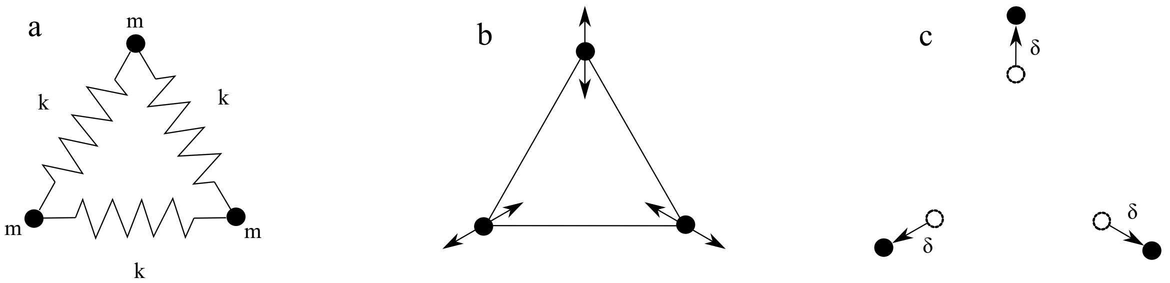
Exercise sheet 7 **Due: Thursday December 17 at 24:00**

1 Planar motion of a triatomic planar molecule



Consider three atoms that are confined to move in the plane ($\mathbb{R}^2$). The three atoms have equal mass m and interact via harmonic forces with equal spring constants k. At equilibrium, the atoms form an equilateral triangle (see figure a).

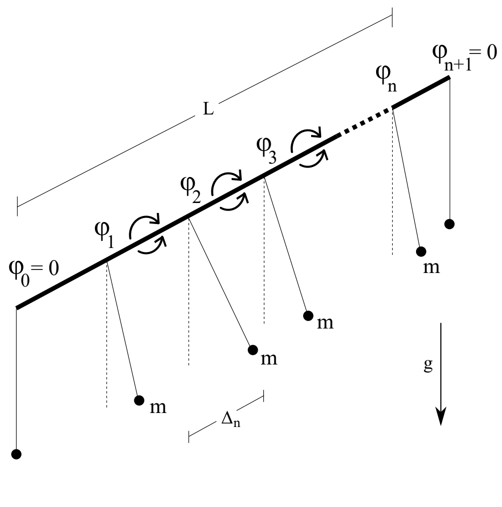
a) *How many independent normal modes are there in this system?* **(2 points)**

b) For a collection of particles that interact harmonically, the zero modes correspond to motions where the particles move collectively as a rigid body. In other words, all the distances between the particles are fixed, but they move jointly. *Without writing down the equations of motion, argue how many zero modes there are for this system.* **(3 points)**

c) One of the normal modes corresponds to a symmetric stretch of the molecule, where it expands and contracts (see figure b). *Determine the frequency of this mode in terms of k and m.*

Hint: One possibility to solve this problem would be to write down the matrix of the eigenvalue problem and find the eigenvalue as we did on exercise sheet 6. However, there is a much quicker way. The motion of all three atoms can be described by the deviation δ from the equilibrium (see figure c). In order to determine the frequency, we can determine the effective equation of motion $M\ddot{\delta} + K\delta = 0$, where M and K are the effective mass and spring constants. How much mass moves when δ changes? The potential energy of the springs are determined by the distance between each pair of atoms. How many such pairs are there? How much does the distance within each pair change when δ changes? **(3 points)**

2 Continuum limit of a chain of mathematical pendula



In the lecture we have seen that one can obtain the wave equation as a continuum limit of a chain of coupled harmonic oscillators. Here we shall obtain the analogous continuum limit for a sequence of coupled mathematical pendula.

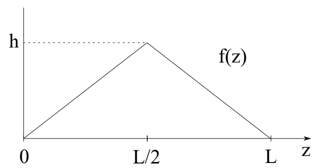
Consider a collection of n pendula, where each has length l and mass m_n, and where φ_j is the angle of the deviation from the vertical downwards direction of pendulum j. Each mass is affected by gravity. Moreover, each pair of neighboring pendula is coupled via a torsion bar, with corresponding potential energy $k_n(\varphi_j - \varphi_{j+1})^2$. We furthermore add fixed endpoints at the angles $\varphi_0 = 0$ and $\varphi_{n+1} = 0$. The resulting equation of motion for the remaining angles $\varphi_1, \ldots, \varphi_n$ is given by

$$m_n l^2 \ddot{\varphi}_j = k_n(\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}) - m_n g l \sin\varphi_j, \quad j = 1, \ldots, n. \tag{1}$$

Like in the lecture, consider a sequence of more and more dense collections of pendula, where we increase the number of pendula n, but keep the total length L fixed. The distance between two neighboring pendula is thus $\Delta_n = \frac{L}{n}$. By varying m_n and k_n in a suitable way (like in the lecture) with increasing number of pendula n (while we regard g and l as fixed) we obtain a linear density ρ and torsional elasticity E. *Analogously to what we did in the lecture, find a limiting wave-equation of (1) as n goes to infinity.* **(5 points)**

3 Playing guitar

In the lecture you discussed a discrete model of a string, and derived the wave-equation as a continuum limit. Here, we will consider the wave-equation itself. As mentioned in the lecture, the equation of motion of a string is given by[1]



$$\left[\frac{d^2}{dt^2} - \frac{E}{\rho}\frac{d^2}{dz^2}\right] d(z,t) = 0, \tag{2}$$

where $d(z)$ is the deviation from equilibrium at position z, E is the elasticity, and ρ is the density.

a) *Confirm that $d(z,t) = e^{ikz - i\omega_k t}$ is a solution to (2) if $\omega_k^2 = \frac{E}{\rho}k^2$.* **(1 point)**

b) Suppose that the string is fixed at the equilibrium position at the endpoints $z = 0$ and $z = L$, i.e.,

$$d(0,t) = 0, \quad d(L,t) = 0, \quad \forall t \in \mathbb{R}. \tag{3}$$

Show that for suitable values of k, the functions

$$d_k(z,t) = e^{\pm itk\sqrt{\frac{E}{\rho}}} v_k(z), \quad v_k(z) = \sqrt{\frac{2}{L}}\sin(kz) \tag{4}$$

are solutions to (2) and (3). Determine the allowed values of k. **(3 points)**

Remark: The collection of functions v_k in (4) form a set of orthonormal mode-functions (for the allowed values of k).

c) Suppose that you pluck the string in the middle. We model this as the function $d(z,0) = f(z)$ in the figure above. We wish to find to what extent each eigenmode of the string gets excited by this. *Find the expansion coefficients of f with respect to the eigenmodes v_k in b).* **(3 points)**

[1]Strictly speaking, this is the equation for transversal vibrations of the string, i.e., motion orthogonal to the string. There could also be longitudinal vibrations, i.e., motion in the direction of the string.