Classical Mechanics

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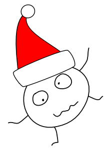
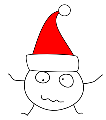
Exercise sheet 8 Due: Thursday January 7 at 24:00



1 The Beltrami identity

Suppose that a function $u(t)$ satisfies the Euler-Lagrange equation

$$\frac{\partial L}{\partial u} = \frac{d}{dt}\frac{\partial L}{\partial \dot{u}} \tag{1}$$



for a Lagrangian $L(t, u, \dot{u})$. In this exercise we shall derive a general relation, sometimes referred to as the Beltrami identity, that is valid for Lagrangians that do not explicitly depend on t.

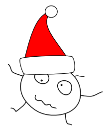
a) To start with, we allow the Lagrangian L to depend on t, as well as on u and $\dot{u}$. *Use the chain rule in order to express the total derivative* $\frac{d}{dt}L$ *in terms of* $\frac{\partial L}{\partial t}$, $\dot{u}$ *and* $\ddot{u}$.

Hint: Keep in mind that $\dot{u} = \frac{du}{dt}$ and $\ddot{u} = \frac{d^2u}{dt^2}$. **(1 point)**

b) *Use the product rule on* $\frac{d}{dt}\left(\dot{u}\frac{\partial L}{\partial \dot{u}}\right)$. **(1 point)**

c) *Combine the results in a) and b) with the Euler-Lagrange equation* (1) *to obtain*

$$\frac{d}{dt}\left(L - \dot{u}\frac{\partial L}{\partial \dot{u}}\right) = \frac{\partial L}{\partial t}.$$



(2 points)

d) Assume that $\frac{\partial L}{\partial t} = 0$. *Show that it follows that there exists a constant C such that*

$$L - \dot{u}\frac{\partial L}{\partial \dot{u}} = C.$$

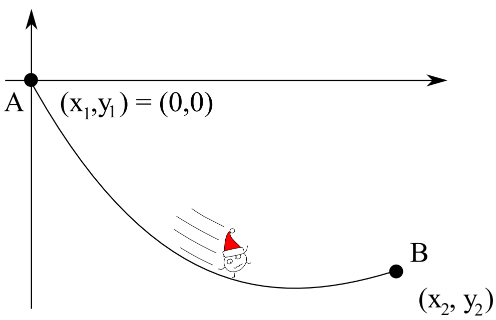
This is the Beltrami-identity.

(2 points)

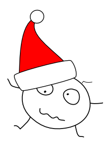
2 Sliding as fast as possible: Brachistochrone

In the lecture we discussed the brachistrochrone as the shape of the curve that give the fastest slide under the influence of gravity. We also learned that we can find the brachistochrone as the solution of the Euler-Lagrange equation corresponding to the Lagrangian

$$L[y, y'] = \sqrt{\frac{1 + y'^2}{2y}}. \tag{2}$$



Here we assume that the initial point A is at $(x_1, y_1) = (0, 0)$ and that the initial velocity is zero. The final point B is at some position (x_2, y_2). We moreover put $g = 1$ and the mass to $m = 1$.



a) *Show that there exists a constant C such that the solution y to the Euler-Lagrange equation corresponding to (2) satisfies*

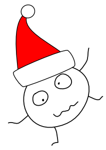
$$\frac{1}{2C^2 y} = 1 + y'^2. \tag{3}$$

(2 points)

b) *Show that*

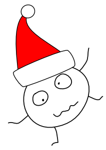
$$\begin{aligned} x =& A(\theta - \sin\theta), \\ y =& A(1 - \cos\theta), \end{aligned} \tag{4}$$

satisfies (3) for some suitable choice of A. **(3 points)**



Remark: Equation (4) describes a cycloid. By following a point on the circumference of a wheel as it rolls, we trace out a cycloid.

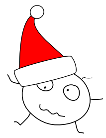
3 Change of variables in the Lagrangian



In the lecture we spent some time on the role of coordinate transformations, and at first sight it might not be clear why this is important. However, coordinate transformations can be a very useful tool for simplifying, and even solving, the dynamics of a system.

Imagine a single particle that is confined to move in the plane ($\mathbb{R}^2$), and is affected by a potential $U(x,y) = \alpha x^2 y^2$, for some positive constant $\alpha > 0$ (and x, y are standard Cartesian coordinates). More exotically, we imagine that the mass of the particle depends on the position[1], such that the mass is $m(x,y) = m_0(x^2 + y^2)$. Apart from the varying mass, the particle has the standard kinetic energy, which results in the Lagrangian

$$L(x,y,\dot{x},\dot{y}) = \frac{m_0}{2}(x^2 + y^2)(\dot{x}^2 + \dot{y}^2) - \alpha x^2 y^2. \tag{5}$$



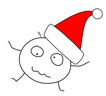
a) *Obtain the EL-equations for the Lagrangian in (5).*

(2 points)

b) Look at the equations that you obtained in a). Firstly, imagine that you would try to solve these equations. Secondly, imagine that you would make the change of variables

$$\begin{aligned} u =& x^2 - y^2, \\ v =& 2xy. \end{aligned} \tag{6}$$

directly in these equations. Take a minute or two to contemplate how horrible this would be, and how profoundly grateful you are that these are *not* tasks in this exercise.[2]



(0 points)

c) *Express the Lagrangian (5) in terms of the new coordinates u and v described in (6). Does this Lagrangian have a cyclic coordinate? If so, what is the corresponding conserved quantity?*

Hint: What is $\dot{u}^2 + \dot{v}^2$? **(3 points)**

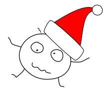
d) *Use the Lagrangian that you obtained in c) in order to find the equations of motions with respect to the variables u, v.* **(2 points)**

e) *Solve the equations of motion that you obtained in d).* **(2 points)**

[1] It is unclear how this would happen, but we can imagine it anyway.

[2] A reasonable manner to express your gratitude would be to draw a very large Å on a wall in your home, and bow towards it three times a day during the entire Christmas break.

4 Gold star exercise: Generalized Euler-Lagrange equation



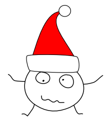
This problem gives no points! However, if you feel uncertain about the theory behind the Euler-Lagrange equations, then this is the exercise for you!

Consider the functional

$$\mathcal{S}[x] = \int_{t_i}^{t_f} f(t, x, \dot{x}, \ddot{x}) dt,$$

i.e., compared to the standard case, we have the additional dependence on $\ddot{x}$. *Show that if $x(t)$ is a stationary point of $\mathcal{S}[x]$, then it satisfies*

$$\frac{d^2}{dt^2}\left(\frac{\partial f}{\partial \ddot{x}}\right) - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{\partial f}{\partial x} = 0.$$



This requires some boundary conditions. What are these boundary conditions?

Hint: The standard E-L equation can be obtained by using the boundary conditions that the values of $x(t_i)$ and $x(t_f)$ are fixed. How could this be generalized? Look up how the derivation of the standard Euler-Lagrange equations works, and generalize that derivation.

(0 points)