Classical Mechanics

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Exercise sheet 11   Due: Thursday January 20 at 24:00

1 Calculating with Poisson brackets

For two functions $f(q_1,\ldots,q_N,p_1,\ldots,p_N,t)$ and $g(q_1,\ldots,q_N,p_1,\ldots,p_N,t)$, the Poisson bracket is defined as[1]

$$\{f,g\} = \sum_{n=1}^{N}\left(\frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n} - \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n}\right). \tag{1}$$

The Poisson bracket satisfies some simple relations, such as

$$\{p_k,q_j\} = -\{q_j,p_k\} = \delta_{kj}, \quad \{q_j,q_k\} = 0, \quad \{p_j,p_k\} = 0.$$

When calculating with Poisson brackets it is often enough to make use of these relations, without having to think about the definition (1), as we shall see in this exercise.

a) *Show that*

$$\{q_j,p_k^n\} = -np_k^{n-1}\delta_{jk}, \quad \{p_j,q_k^n\} = nq_k^{n-1}\delta_{jk}, \quad n = 1,2,3,\ldots$$

Hint: This is efficiently shown via an induction proof. **(4 points)**

b) The angular momentum of a particle with position $\vec{q}$ and momentum $\vec{p}$, is given by $\vec{L} = \vec{q}\times\vec{p}$. For $\vec{L} = (L_1,L_2,L_3)$, one can write $L_j = \sum_{kl}\epsilon_{jkl}q_kp_l$, where ϵ_{jkl} is the Levi-Civita symbol[2].

Show that

$$\{L_j,q_n\} = \sum_k \epsilon_{njk}q_k, \quad \{L_j,p_n\} = \sum_l \epsilon_{njl}p_l, \quad j,n = 1,2,3,$$

and

$$\{L_j,\vec{q}^2\} = 0, \quad \{L_j,\vec{p}^2\} = 0, \quad j = 1,2,3.$$

Hint: Note that $(\vec{a}\times\vec{b})_j = \sum_{kl}\epsilon_{jkl}a_kb_l$ and $\vec{a}\times\vec{a} = 0$. Note also that whenever you permute two indices in ϵ_{jkl}, then it changes sign, e.g., $\epsilon_{jkl} = -\epsilon_{kjl} = \epsilon_{klj}$.

(5 points)

c) Consider the rotation

$$\Phi^{(s)} = \begin{bmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and define the rotated vectors

$$\vec{q}(s) = \begin{bmatrix} q_1(s) \\ q_2(s) \\ q_3(s) \end{bmatrix} = \Phi^{(s)}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \quad \vec{p}(s) = \begin{bmatrix} p_1(s) \\ p_2(s) \\ p_3(s) \end{bmatrix} = \Phi^{(s)}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

[1] The definition of the Poisson bracket occurs in two variations that differ in the choice of the overall sign. This choice affects the sign of the bracket $\{p_k,q_l\}$.

[2] The Levi-Civita symbol in three dimensions is $\epsilon_{jkl} = \begin{cases} 1 & (jkl)\in\{(123),(312),(231)\} \\ -1 & (jkl)\in\{(213),(321),(132)\} \\ 0 & \text{else} \end{cases}.$

Around which axis is $\Phi^{(s)}$ a rotation? Determine $\frac{d}{ds}\vec{q}|_{s=0}$ and $\frac{d}{ds}\vec{p}|_{s=0}$, and compare with $\{L_3, \vec{q}(0)\}$ and $\{L_3, \vec{p}(0)\}$.

(4 points)

Remark: This exercise indicates that angular momentum is the generator of rotations.

2 Conserved quantities via Poisson brackets

This exercise exemplifies that Poisson brackets can be used in order to identify conserved quantities.

Suppose that we have two particles of mass m that move in three-dimensional space ($\mathbb{R}^3$), and interact with each other via a quadratic potential. This can be described with the following Hamilton function:

$$H(\vec{q}_1, \vec{q}_2, \vec{p}_1, \vec{p}_2) = \frac{1}{2m}\vec{p}_1^2 + \frac{1}{2m}\vec{p}_2^2 - \alpha\|\vec{q}_1 - \vec{q}_2\|^2.$$

Let $\vec{L}^{(1)} = \vec{q}_1 \times \vec{p}_1$ and $\vec{L}^{(2)} = \vec{q}_2 \times \vec{p}_2$ be the angular momentum vectors of particle 1 and 2, respectively.

a) *Show that*

$$\left\{H, L_j^{(1)} + L_j^{(2)}\right\} = 0, \quad j = 1, 2, 3. \tag{2}$$

Hint: There are various observations that make the derivation quicker. For example, $\{f(\vec{q}_1, \vec{p}_1) + f(\vec{q}_2, \vec{p}_2), g(\vec{q}_1, \vec{p}_1) + g(\vec{q}_2, \vec{p}_2)\} = \{f(\vec{q}_1, \vec{p}_1), g(\vec{q}_1, \vec{p}_1)\} + \{f(\vec{q}_2, \vec{p}_2), g(\vec{q}_2, \vec{p}_2)\}$. The relations that you proved in exercise 1b) could be useful. Recall that the cross-product can be written $(\vec{a} \times \vec{b})_j = \sum_{kl} \epsilon_{jkl} a_k b_l$, where ϵ_{jkl} is the Levi-Civita symbol.

(5 points)

b) *In more physical terms, what is it that you have proved with equation (2)?* **(2 points)**