Classical Mechanics

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Exercise sheet 12 Due: Thursday January 27 at 24:00

1 Solving the equations of motion via canonical transformations

In this exercise we use canonical transformations in order to solve the equations of motion of the following Hamiltonian,

$$H(q,p) = \frac{1}{2}p^2q^4 + \frac{1}{2q^2}. \tag{1}$$

The strategy will be to find a canonical transformation that turns (1) into a harmonic oscillator. By solving the latter, we can then work backwards in order to find an explicit formula for the evolution caused by (1). The question is how to find that first canonical transformation. Here we shall create a family of canonical transformations, and hope that it is big enough to achieve what want.[1]

a) Since we want to turn some combinations of powers of q and p (in (1)) into some other powers of Q and P of the harmonic oscillator (in (3)), one idea would be to try some transformations that combine powers of q and p. The problem is that arbitrary expressions would generally not result in canonical transformations, so we have to determine which combinations of powers yield valid canonical transformations.

Consider the family of transformations from (q,p) to (Q,P) defined by[2]

$$P = \alpha p^\beta q^\gamma, \quad Q = q^\delta, \tag{2}$$

where α, β, γ and δ are constants. *What are the conditions on α, β, γ and δ for (2) to be a canonical transformation?*

Hint: Recall the characterization of canonical transformations in terms of Poisson brackets, and the definition of the Poisson bracket.

(3 points)

b) *Use the result from a) to find a canonical transformation from (q,p) to (Q,P) that transforms the Hamiltonian in (1) into the Hamilton function of the Harmonic oscillator*

$$H(Q,P) = \frac{1}{2}P^2 + \frac{1}{2}Q^2. \tag{3}$$

(3 points)

c) *Use the result in b) in order to solve the equations of motion generated by (1).*

(3 points)

2 Generating functions for canonical transformations

Generating functions can be used to find canonical transformations from coordinates $(\vec{q},\vec{p})$ to a new set of coordinates $(\vec{Q},\vec{P})$. One example is that we have a function $F(q,Q)$ of the old coordinate

[1] There is no guarantee that this works. Basically, it is like shooting from the hip; if we are lucky we hit.

[2] As I said, this is a wild guess. One could of course also take larger families of transformations, with more free parameters, but let's not get too extreme.

q and the new coordinate Q. Such a function does implicitly define a canonical transformation between (q, p) and (Q, P) via the two equations

$$p = \frac{\partial F}{\partial q}, \quad P = -\frac{\partial F}{\partial Q}. \tag{4}$$

a) Consider the function

$$F(q, Q) = \frac{m\omega}{2} q^2 \frac{1}{\tan Q},$$

where m and ω are some constants. *Use the relations (4) in order to express q and p as functions of Q and P.*

Hint: The relations (4) give p and P as functions of q and Q, and you have to transform these so that you obtain q and p as functions of Q and P. Do not worry about whether the square roots are well defined, or about the sign of the roots. **(3 points)**

b) Consider the Hamilton function for the harmonic oscillator

$$H(q, p) = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} q^2.$$

Express H in terms of the new variables Q and P. What is the solution of the corresponding equations of motion of Q and P? Transform the solution back to the original (q, p) and thus obtain the solutions to the equations of motion of the harmonic oscillator. **(3 points)**

3 Generating functions for canonical transformations again

In the previous exercise we considered generating functions of the type $F(q, Q)$. However, one can also use generating functions of the form $F_2(q, P)$, $F_3(p, Q)$, or $F_4(p, P)$. As an example we are here going to consider generating functions of the form $F_3(p, Q)$. Such functions define canonical transformations between (q, p) and (Q, P) via

$$q = -\frac{\partial F_3}{\partial p}, \quad P = -\frac{\partial F_3}{\partial Q}. \tag{5}$$

Note the difference in signs compared to (4)!

a) Consider the function

$$F_3(p, Q) = -(e^Q - 1)^2 \tan p.$$

By using (5), determine Q and P as functions of q and p.

Hint: As in the previous exercise, do not worry about square roots (or logarithms) being well defined, or which branches to take. **(2 points)**

b) *Confirm, by using Poisson brackets, that the functions $Q(q, p)$ and $P(q, p)$ obtained in a) define a canonical transformation from (q, p) to (Q, P).* **(3 points)**