Classical Mechanics

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Exercise sheet 4 **Due: Thursday November 11 at 24:00**

1 Application of Kepler's laws

Kepler's first law states that the planets move along ellipses around the sun[1]. One way to describe the orbit is by using polar coordinates, where the radius r depends on the angle θ as

$$r(\theta) = \frac{p}{1+\epsilon\cos\theta} \tag{1}$$

where ϵ is the eccentricity of the ellipse, and p gives the size of the orbit. Kepler's second law says that the line joining the planet and the sun sweeps out equal areas during equal intervals of time. Kepler's third law states the length a of the semimajor axis of the ellipse relates to the period T of the orbit as

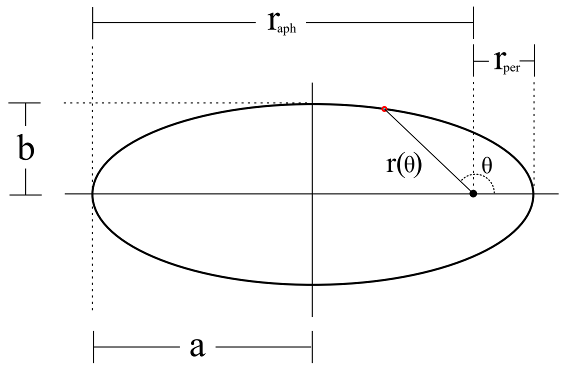

Abbildung 1: The semimajor axis a and the semiminor axis b of the ellipse. The shortest distance r_{per} to the sun (perihelion), and the longest distance r_{aph} (aphelion).

$$\frac{a^3}{T^2} = \frac{G(M+m)}{(2\pi)^2},$$

where M and m are the mass of the sun and the planet, respectively. Note that $G \approx 6.67 * 10^{-11} Nm^2/kg^2$.

a) Before landing, Apollo 11 was put in orbit around the moon. The mass of of Apollo 11 was 9979 kg and the period of the orbit was 119 min. The maximum and minimum distances from the center of the moon were 1861 km and 1838 km, respectively. *Use these data to estimate the mass of the moon.* **(2 points)**

b) Halley's comet moves in an elliptic orbit around the sun, with a period of 76 years. The eccentricity is $\epsilon = 0.97$. The mass of the sun is about $M = 2.0 * 10^{30} kg$. *Use these data to determine the distance from the sun at perihelion (when the comet is the closest to the sun) and aphelion (when it is the most far away).*

Hint: We can ignore the mass of the comet compared to the mass of the sun. **(3 points)**

[1] Kepler's laws is not only applicable to things orbiting around the sun, but can be applied also to other constellations. Strictly speaking, Eq. (1) describes the motion around the center of mass of the two bodies. However, in our case $M \gg m$ and thus the center of mass is approximately the position of the heavier body.

c) *What is the ratio of the speed of Halley's comet when it is in perihelion compared with when it is in aphelion?* It is useful to keep in mind that $r^2 \frac{d\theta}{dt} =$ constant (which comes from the conservation of angular momentum).

Hint: What is the radial speed at perihelion and aphelion? How does the tangential speed relate to the angular speed and r?

(3 points)

Comment: The purpose of this exercise is to demonstrate some uses of Kepler's laws.

2 Maximum angular momentum for a circular orbit

As we have seen in the lecture, two-body problems (in $\mathbb{R}^3$) can be simplified quite dramatically, resulting in an effective one-dimensional motion of a fictitious particle. Consider a two-body problem, with reduced mass μ, where the interaction potential can be expressed in terms of the relative coordinate $\vec{r}$ as

$$U(r) = -U_0 e^{-\lambda^2 r^2}, \quad r = \|\vec{r}\|, \quad U_0 > 0, \quad \lambda > 0. \tag{2}$$

a) *Determine the effective potential U_{eff} corresponding to (2) for a given magnitude of the angular momentum $\|\vec{L}\| = \ell$.*

(1 point)

b) *Show that the radius r_0 of any circular orbit has to satisfy the equation*

$$\ell^2 = A r_0^B e^{-\lambda^2 r_0^2}, \tag{3}$$

and determine the constants $A \geq 0$ and $B \geq 0$. **(2 points)**

c) *Find the largest value of ℓ such that there exists a circular orbit.*

Hint: The right hand side of (3), regarded as a function of r_0, has one single maximum.

(2 points)

3 The Laplace-Runge-Lenz vector

In the lecture we have discussed centrally symmetric potentials. In this exercise we are going to consider the special class of centrally symmetric potentials of the form

$$U(\vec{r}) = -\frac{k}{r}, \quad r = \|\vec{r}\|, \tag{4}$$

for some constant k, where examples are gravity and the Coulomb potential. The question of the dynamics of systems of this type is often referred to as the "Kepler problem".

In the lecture we demonstrated that the angular momentum is conserved for centrally symmetric potentials. Moreover, like for any other conservative force, the energy is conserved. Here we are going to show that for the Kepler problem, i.e., for potentials of the form (4), there exists an additional "accidental" conserved quantity, namely the Laplace-Runge-Lenz vector, which is defined as

$$\vec{A} = \mu \dot{\vec{r}} \times \vec{L} - \mu k \hat{r},$$

where μ is the reduced mass, $\hat{r}$ is the unit vector in the direction of $\vec{r}$, and $\vec{L}$ is the angular momentum. *Show that $\vec{A}$ is conserved, i.e., show that $\frac{d\vec{A}}{dt} = 0$.*

Hint: What is the force in the current case? How does the force relate to $\mu \ddot{\vec{r}}$? Keep in mind that $\vec{L}$ is conserved. Make use of the general relation for cross-products $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$. Show that $\vec{r} \cdot \frac{d\vec{r}}{dt} = r\frac{dr}{dt}$. **(7 points)**