# Classical Mechanics

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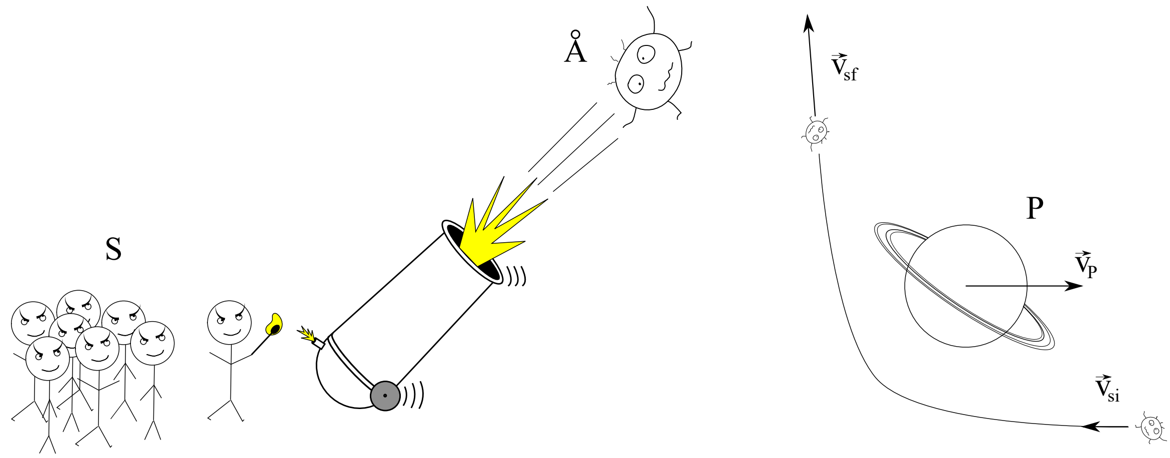
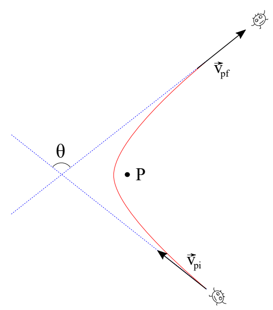
**Exercise sheet 5** **Due: Thursday November 18 at 24:00**

## 1 Shooting someone very far into space




A group of students S want to send an individual Å, who creates exercises in classical mechanics, very far into space. They use a cannon to shoot this object into space[1]. They have enough gunpowder to reach the necessary escape velocity from Earth. However, they want to make sure that Å disappears deeply into interstellar space, but at the same time they want to use as little gunpowder as possible, so that they have more money left for the party afterwards. Some of them have read a Wikipedia-article about a clever technique called the "gravitational slingshot´´ (or "gravity assist maneuver´´, or "swing-by´´). The idea is that one can increase the speed (relative to the sun) of a spacecraft (or an object Å) by letting it pass near a planet in a suitable way. The problem is that the group of students S are not quite sure how all of this works, so you have to help them to understand the general principles.




As mentioned above, the gravitational sling-shot works by letting object Å pass near a planet P. For the first step of the analysis we consider the system in the center of mass frame of Å and P[2]. We assume that the mass $M$ of P is much larger than the mass $m$ of Å. Hence, we can to a very good approximation put the center of mass at the center of P, and the relative mass $\mu$ equal to $m$. The total energy of Å is positive[3] since otherwise Å would be trapped in a periodic orbit around P[4]. Hence, Å follows a hyperbolic orbit ($\epsilon > 1$).

We let $\vec{v}_{pi}$ and $\vec{v}_{pf}$ denote the initial and final velocity of Å very far from P, *in the center of mass frame*. The angle between these two vectors, i.e., the angle $\theta$ such that $\vec{v}_{pi} \cdot \vec{v}_{pf} = v_{pi} v_{pf} \cos\theta$, is referred to as the *scattering angle*.

**a)** *Show that the initial speed $v_{pi} = \|\vec{v}_{pi}\|$ and final speed $v_{pf} = \|\vec{v}_{pf}\|$ are equal.* **(1 point)**

[1] Although they do intend to become dictators (as one can see on their eyebrows) they are still only doing dictator-internships, so they can unfortunately not afford a rocket.

[2] We ignore the influence from the sun.

[3] We follow the convention to put the potential energy to zero infinitely far away from P.

[4] Strictly speaking zero energy would also be allowed. Then, the orbit would be parabolic (and thus $\epsilon = 1$), but we skip that special case here.

**b)** The goal is to determine the difference between the final kinetic energy $E_{\text{kin},f}$ of object Å and and the initial kinetic energy $E_{\text{kin},i}$, *with respect to the reference frame of the sun*. Let $\vec{v}_P$ denote the velocity of planet P (also in the frame of the sun). *Show that*

$$E_{\text{kin},f} - E_{\text{kin},i} = m(\vec{v}_{pf} - \vec{v}_{pi}) \cdot \vec{v}_P.$$

**Hint:** Let $\vec{v}_{si}$ and $\vec{v}_{sf}$ be the initial and final velocities of Å *in the reference frame of the sun*. What is the relation between $\vec{v}_{si}$ and $\vec{v}_{sf}$ and $\vec{v}_{pi}$ and $\vec{v}_{pf}$? **(2 points)**

**c)** To demonstrate the gravitational slingshot one can consider the special case that the initial velocity $\vec{v}_{si}$ of Å, in the frame of the sun, is anti-parallel to $\vec{v}_P$. In other words, Å initially moves in the opposite direction compared to planet P. Hence, we let $\vec{v}_{si} = -s\vec{v}_P$ for $s > 0$. *Show that*

$$E_{\text{kin},f} - E_{\text{kin},i} = m(1+s)\|\vec{v}_P\|^2\Big(1 - \cos\theta\Big),$$

*where $\theta$ is the scattering angle. What does this imply for the relation between $E_{\text{kin},f}$ and $E_{\text{kin},i}$? Where does the energy come from?* **(3 points)**

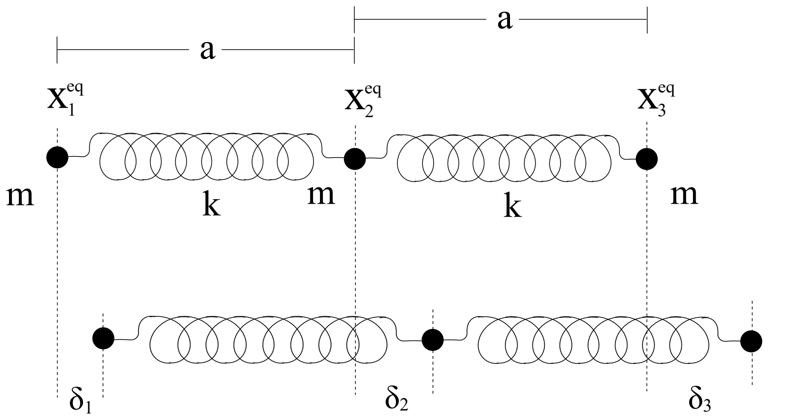
**Hint:** Recall that $\vec{v}_{pi} \cdot \vec{v}_{pf} = v_{pi} v_{pf} \cos\theta$.

## 2 Linear molecule with equal masses

In the lecture we considered the vibrations of $N$ particles with periodic boundary conditions. In this exercise we consider a model of a linear molecule consisting of three atoms, each of mass $m$ that interact according to the potential

$$U(x_1, x_2, x_3) = \frac{k}{2}(x_2 - x_1 - a)^2 + \frac{k}{2}(x_3 - x_2 - a)^2,$$

where $k > 0$ and $a > 0$, with open boundary conditions. For the sake of simplicity we here only consider the motion of the atoms along the axis of the molecule.




Recall from the lecture that the motion of particles that interact harmonically can be decomposed into especially simple components, the *normal modes*. For each such normal mode, the particles perform a collective periodic motion with respect to one single frequency.

This molecule is in mechanical equilibrium when the positions of the atoms are such that $x_1 = x_1^{\text{eq}}$, $x_2 = x_2^{\text{eq}}$, and $x_3 = x_3^{\text{eq}}$, where $x_2^{\text{eq}} - x_1^{\text{eq}} = a$ and $x_3^{\text{eq}} - x_2^{\text{eq}} = a$. It is very convenient to introduce new coordinates $\delta_1 = x_1 - x_1^{\text{eq}}$, $\delta_2 = x_2 - x_2^{\text{eq}}$, and $\delta_3 = x_3 - x_3^{\text{eq}}$ that measure the *deviation* from equilibrium. With respect to these new coordinates, we can write the potential as

$$U(\delta_1, \delta_2, \delta_3) = \frac{k}{2}(\delta_2 - \delta_1)^2 + \frac{k}{2}(\delta_3 - \delta_2)^2.$$

**a)** In the lecture you have seen that one can write the equations of motion as

$$\frac{d^2}{dt^2}\vec{\delta} = \boldsymbol{M}\vec{\delta}, \quad \vec{\delta} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}, \tag{1}$$

for some matrix $\boldsymbol{M}$. *Determine the matrix* $\boldsymbol{M}$. **(1 point)**

**b)** In order to find the normal modes one can make the ansatz $\vec{\delta}(t) = e^{i\omega t}\vec{v}$, where $\vec{v}$ is a time-independent vector, and $\omega$ a real number. *Show that this leads to an eigenvalue problem of the form*

$$\lambda\vec{v} = \boldsymbol{M}\vec{v}, \tag{2}$$

*where the eigenvalue $\lambda$ is a function of $\omega$. Determine this function.* **(1 point)**

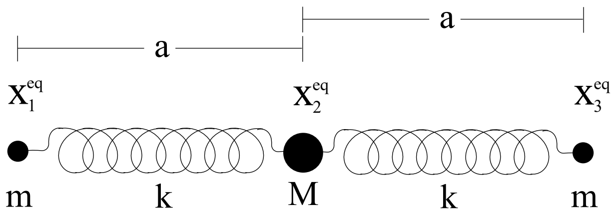
**c)** *Determine the eigenvalues $\lambda$ and eigenvectors $\vec{v}$ of (2). For each eigenvalue $\lambda$, what are the possible values of $\omega$?* **(3 points)**

**d)** *Sketch the motion of the normal modes of the linear molecule. What kind of motion does the zero eigenvalue correspond to? What conservation law is it related to?* **(2 points)**

**e)** *By using the results in c), write down the corresponding solutions to (1). How can you combine these solutions in order to guarantee that the resulting function is real-valued (and thus describes an actual physical motion)?* **(2 points)**

## 3 Linear molecule with different masses

Let us now change the problem in exercise 2 such that the middle particle (the one with equilibrium position $x_2^{\text{eq}}$) could have a mass $M$ different from $m$ (but we let the interaction potential be the same as before, and we still restrict the motion to be one-dimensional).




**a)** *Show that the equations of motion can be written as*

$$\frac{d^2}{dt^2}\vec{\delta} = \boldsymbol{W}\vec{\delta},$$

*and determine the matrix* $\boldsymbol{W}$. **(1 point)**

**b)** *Determine eigenvalues of* $\boldsymbol{W}$*, and the corresponding frequencies* $\omega$. **(2 points)**

**c)** The spectrum of $CO_2$ might determine the fate of humanity. The calculation in this exercise can already lead to some predictions about its properties. We cannot deduce the spring constant k, as this requires a quantum mechanical model. However, note that the ratio of the frequencies is independent of k. Thus our simple theory suggests a value for the ratio of some of the frequencies absorbed by atmospheric $CO_2$. *Compute the ratio, and compare it to experimental results.*

**Hint:** A suitable source could be "NIST's Tables of Molecular Vibrational Frequencies": https://nvlpubs.nist.gov/nistpubs/Legacy/NSRDS/nbsnsrds39.pdf

Look for the frequencies of the "stretching" modes – there is also a "bending" mode, which we have not considered here. **(2 points)**