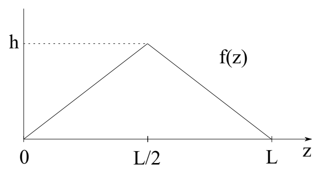
Classical Mechanics

David Gross, David Wierichs, Markus Heinrich, Johan Åberg

Exercise sheet 6 Due: Thursday November 25 at 24:00

1 Playing guitar



In the lecture you discussed a discrete model of a string, and derived the wave-equation as a continuum limit. Here, we will consider the wave-equation itself. As mentioned in the lecture, the equation of motion of a string is given by

$$\left[\frac{\partial^2}{\partial t^2} - \frac{E}{\rho}\frac{\partial^2}{\partial z^2}\right] d(z,t) = 0, \tag{1}$$

where $d(z)$ is the deviation from equilibrium at position z, E is the elasticity, and ρ is the density.

a) *Confirm that $d(z,t) = e^{ikz - i\omega_k t}$ is a solution to (1) if $\omega_k^2 = \frac{E}{\rho}k^2$.* **(1 point)**

b) Suppose that the string is fixed at the equilibrium position at the endpoints $z = 0$ and $z = L$, i.e.,

$$d(0,t) = 0, \quad d(L,t) = 0, \quad \forall t \in \mathbb{R}. \tag{2}$$

Show that for suitable values of k, the functions

$$d_k(z,t) = e^{\pm itk\sqrt{\frac{E}{\rho}}} v_k(z), \quad v_k(z) = \sqrt{\frac{2}{L}}\sin(kz) \tag{3}$$

are solutions to (1) and (2). Determine the allowed values of k. **(3 points)**

Remark: The collection of functions v_k in (3) form a set of orthonormal mode-functions (for the allowed values of k).

c) Suppose that you pluck the string in the middle. We model this as the function $d(z,0) = f(z)$ in the figure above. We wish to find to what extent each eigenmode of the string gets excited by this. *Find the expansion coefficients of f with respect to the eigenmodes v_k in b).* **(3 points)**

2 The Beltrami identity

Suppose that a function $u(t)$ satisfies the Euler-Lagrange equation

$$\frac{\partial L}{\partial u} = \frac{d}{dt}\frac{\partial L}{\partial \dot{u}} \tag{4}$$

for a Lagrangian $L(t, u, \dot{u})$. In this exercise we shall derive a general relation, sometimes referred to as the Beltrami identity, that is valid for Lagrangians that do not explicitly depend on t.

a) To start with, we allow the Lagrangian L to depend on t, as well as on u and $\dot{u}$. *Use the chain rule in order to express the total derivative $\frac{d}{dt}L$ in terms of $\frac{\partial L}{\partial t}$, $\dot{u}$ and $\ddot{u}$.*

Hint: Keep in mind that $\dot{u} = \frac{du}{dt}$ and $\ddot{u} = \frac{d^2u}{dt^2}$. **(1 point)**

b) *Use the product rule on $\frac{d}{dt}\left(\dot{u}\frac{\partial L}{\partial \dot{u}}\right)$.* **(1 point)**

c) *Combine the results in a) and b) with the Euler-Lagrange equation (4) to obtain*

$$\frac{d}{dt}\left(L - \dot{u}\frac{\partial L}{\partial \dot{u}}\right) = \frac{\partial L}{\partial t}.$$

(3 points)

d) Assume that $\frac{\partial L}{\partial t} = 0$. *Show that it follows that there exists a constant C such that*

$$L - \dot{u}\frac{\partial L}{\partial \dot{u}} = C.$$

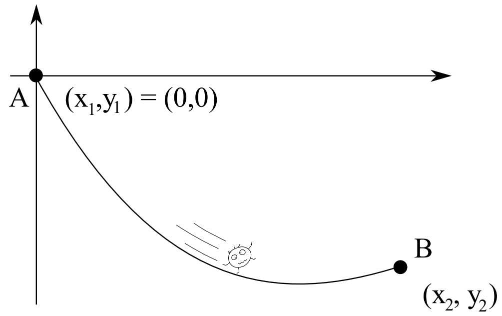
This is the Beltrami-identity.

(2 points)

3 Sliding as fast as possible: Brachistochrone

In the lecture we discussed the brachistrochrone as the shape of the curve that give the fastest slide under the influence of gravity. We also learned that we can find the brachistochrone as the solution of the Euler-Lagrange equation corresponding to the Lagrangian

$$L[y, y'] = \sqrt{\frac{1 + y'^2}{2y}}. \tag{5}$$



Here we assume that the initial point A is at $(x_1, y_1) = (0, 0)$ and that the initial velocity is zero. The final point B is at some position (x_2, y_2). We moreover put $g = 1$ and the mass to $m = 1$.

a) *Show that there exists a constant C such that the solution y to the Euler-Lagrange equation corresponding to (5) satisfies*

$$\frac{1}{2C^2 y} = 1 + y'^2. \tag{6}$$

(3 points)

b) *Show that*

$$\begin{aligned} x =& A(\theta - \sin\theta), \\ y =& A(1 - \cos\theta), \end{aligned} \tag{7}$$

satisfies (6) for some suitable choice of A. **(3 points)**

Remark: Equation (7) describes a cycloid. By following a point on the circumference of a wheel as it rolls, we trace out a cycloid.

4 Gold star exercise: Generalized Euler-Lagrange equation

This problem gives no points! However, if you feel uncertain about the theory behind the Euler-Lagrange equations, then this is the exercise for you!

Consider the functional

$$\mathcal{S}[x] = \int_{t_i}^{t_f} f(t, x, \dot{x}, \ddot{x}) dt,$$

i.e., compared to the standard case, we have the additional dependence on $\ddot{x}$. *Show that if $x(t)$ is a stationary point of $\mathcal{S}[x]$, then it satisfies*

$$\frac{d^2}{dt^2}\left(\frac{\partial f}{\partial \ddot{x}}\right) - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{\partial f}{\partial x} = 0.$$

This requires some boundary conditions. What are these boundary conditions?

Hint: The standard E-L equation can be obtained by using the boundary conditions that the values of $x(t_i)$ and $x(t_f)$ are fixed. How could this be generalized? Look up how the derivation of the standard Euler-Lagrange equations works, and generalize that derivation.

(0 points)