Classical Mechanics

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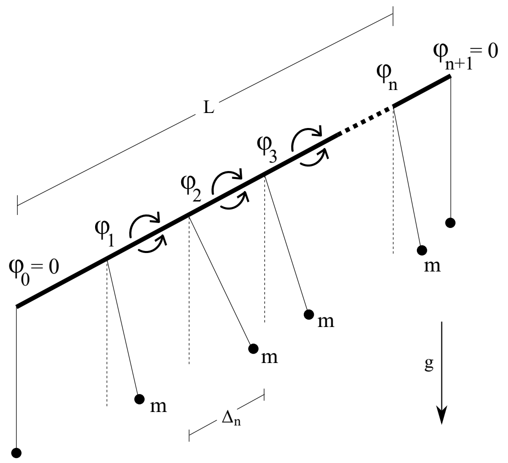
Exercise sheet 7 Due: Thursday December 2 at 24:00

1 Continuum limit of a chain of mathematical pendula



In the lecture we have seen that one can obtain the wave equation as a continuum limit of a chain of coupled harmonic oscillators. Here we shall obtain the analogous continuum limit for a sequence of coupled mathematical pendula.

Consider a collection of n pendula, where each has length l and mass m_n, and where φ_j is the angle of the deviation from the vertical downwards direction of pendulum j. Each mass is affected by gravity. Moreover, each pair of neighboring pendula is coupled via a torsion bar, with corresponding potential energy $k_n(\varphi_j - \varphi_{j+1})^2$. We furthermore add fixed endpoints at the angles $\varphi_0 = 0$ and $\varphi_{n+1} = 0$. The resulting equation of motion for the remaining angles $\varphi_1, \dots, \varphi_n$ is given by

$$m_n l^2 \ddot{\varphi}_j = k_n(\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}) - m_n g l \sin\varphi_j, \quad j = 1, \dots, n. \tag{1}$$

Like in the lecture, consider a sequence of more and more dense collections of pendula, where we increase the number of pendula n, but keep the total length L fixed. The distance between two neighboring pendula is thus $\Delta_n = \frac{L}{n}$. By varying m_n and k_n in a suitable way (like in the lecture) with increasing number of pendula n (while we regard g and l as fixed) we obtain a linear density ρ and torsional elasticity E.

Analogously to what we did in the lecture, find a limiting wave-equation of (1) as n goes to infinity.

(4 points)

2 Block and wedge

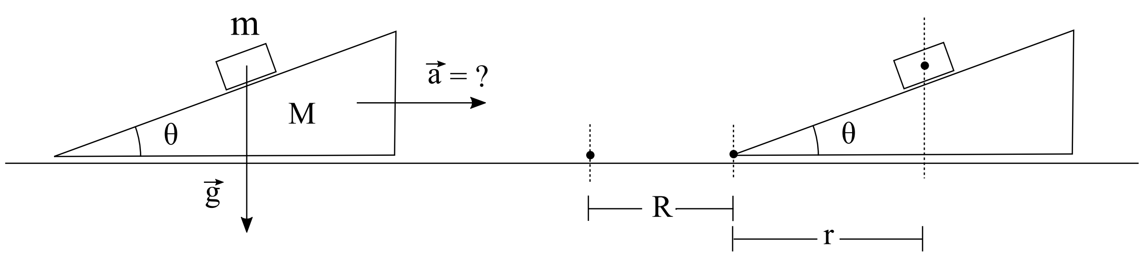


Figure 1: A wedge of mass M can slide without friction along a horizontal surface. On the wedge there is a block of mass m that slides along the surface of the wedge without friction. The block is affected by gravity. The coordinate r is the horizontal distance from the edge of the wedge to the center of mass of the block. R is the distance from the edge of the wedge to some reference point on the plane.

A wedge can slide without friction along a floor, and on the wedge there is a block that also can slide without friction. Imagine that we initially hold both the block and the wedge still, and then suddenly release them.

a) *Derive the Lagrange function for the wedge and block with respect to the coordinates r and R described in the figure.*[1] **(2 points)**

b) *Use the Lagrange function to obtain the Euler-Lagrange equations.* **(2 points)**

c) *Use the Euler-Lagrange equations to find the acceleration of the wedge.* **(2 points)**

d) When you solved b) you may have noted that $\frac{\partial L}{\partial R} = 0$.[2] By the Euler-Lagrange equation it follows that $\frac{d}{dt}\frac{\partial L}{\partial \dot{R}} = 0$. This means that $\frac{\partial L}{\partial \dot{R}}$ is a time-independent quantity, i.e., it is conserved. *What conserved quantity does $\frac{\partial L}{\partial \dot{R}}$ correspond to? Can you explain, in terms of forces, why this quantity is conserved?* **(2 points)**

Comment: As a comparison, you could try to find the equations of motion via the standard Newtonian methods. You will most likely discover that it is much more straightforward to use the Lagrangian approach.

3 Change of variables in the Lagrangian

In the lecture we spent some time on the role of coordinate transformations, and at first sight it might not be clear why this is important. However, coordinate transformations can be a very useful tool for simplifying, and even solving, the dynamics of a system.

Imagine a single particle that is confined to move in the plane ($\mathbb{R}^2$), and is affected by a potential $U(x, y) = \alpha x^2 y^2$, for some positive constant $\alpha > 0$ (and x, y are standard Cartesian coordinates). More exotically, we imagine that the mass of the particle depends on the position[3], such that the mass is $m(x, y) = m_0(x^2 + y^2)$. Apart from the varying mass, the particle has the standard kinetic energy, which results in the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \frac{m_0}{2}(x^2 + y^2)(\dot{x}^2 + \dot{y}^2) - \alpha x^2 y^2. \tag{2}$$

a) *Obtain the EL-equations for the Lagrangian in (2).*

(2 points)

b) Look at the equations that you obtained in a). Firstly, imagine that you would try to solve these equations. Secondly, imagine that you would make the change of variables

$$\begin{aligned} u &= x^2 - y^2, \\ v &= 2xy. \end{aligned} \tag{3}$$

directly in these equations. Take a minute or two to contemplate how horrible this would be, and how profoundly grateful you are that these are *not* tasks in this exercise.

(0 points)

c) *Express the Lagrangian (2) in terms of the new coordinates u and v described in (3). Does this Lagrangian have a cyclic coordinate? If so, what is the corresponding conserved quantity?*

Hint: What is $\dot{u}^2 + \dot{v}^2$? **(2 points)**

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[1] The beauty of the Lagrange method is that one can use whichever coordinates one wants. However, let us nevertheless settle for this choice of coordinates so that we do not drive the poor tutors to the brink of tears. (They would have to check each new derivation for each and every choice of coordinates that you could come up with.)

[2] When $\frac{\partial L}{\partial R} = 0$ the coordinate R is often referred to as being 'cyclic'.

[3] It is unclear how this would happen, but we can imagine it anyway.

d) *Use the Lagrangian that you obtained in c) in order to find the equations of motion with respect to the variables u, v.* **(2 points)**

e) *Solve the equations of motion that you obtained in d).* **(2 points)**