Classical Mechanics

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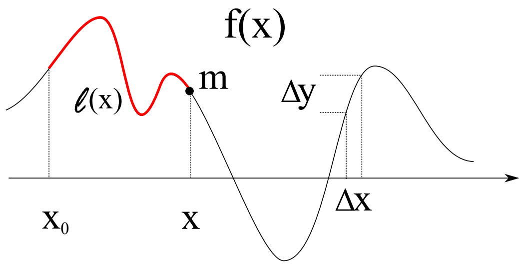
Exercise sheet 8 **Due: Thursday December 9 at 24:00**

1 Change of variable for a constrained particle

Imagine a particle that is constrained to move along a curve (x, y) where $y = f(x)$ for some function f [1]. The particle is otherwise not affected by any potentials.



a) *What is the Lagrangian of this system, in terms of the coordinate x?* **(2 points)**

b) *Derive the Euler-Lagrange equation.* **(2 points)**

c) *Find the general expression for the length $\ell(x)$ of the curve, from the point $(x_0, f(x_0))$ to $(x, f(x))$, for some arbitrary but fixed x_0.*

Hint: The answer is an integral. Think of the interval $[x, x + \Delta x]$ for a small Δx. What is the approximate size of Δy? What is the length of the line that connects the point (x, y) to $(x + \Delta x, y + \Delta y)$? **(2 points)**

d) *Make a change of variables to the new coordinate ℓ in the Lagrangian. What is the Euler-Lagrange equation in this case? Find the general solution.* **(3 points)**

Comment: This exercise illustrates that one can treat very general constraints with Lagrangian mechanics. It also demonstrates that we sometimes can make the equations of motion very simple by a suitable choice of coordinates.

2 Bead on a hoop

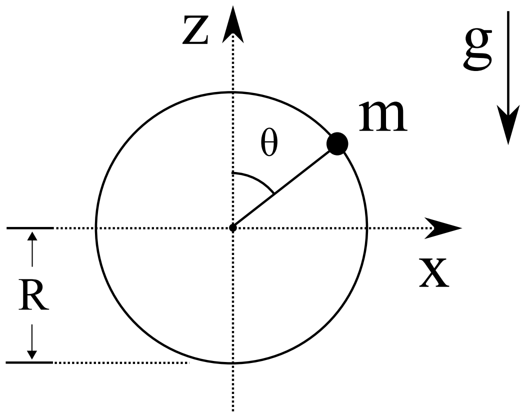
We consider a bead of mass m that can slide without friction on a hoop that is oriented vertically, so that the mass m is affected by the constant gravitational acceleration g.

a) *Derive the Lagrangian in terms of the angle θ, and obtain the corresponding Euler-Lagrange equation.* **(3 points)**

b) Suppose that the bead at a given moment of time is at angle θ and has the angular speed $\dot{\theta}$. In order to keep the bead at the constant radius R, the hoop must exert a normal force $\vec{F}_N = F_N \hat{r}$ on the bead, where $\hat{r}$ is the unit radial vector. *Show that*

$$F_N = mg\cos\theta - mR\dot{\theta}^2. \quad (1)$$

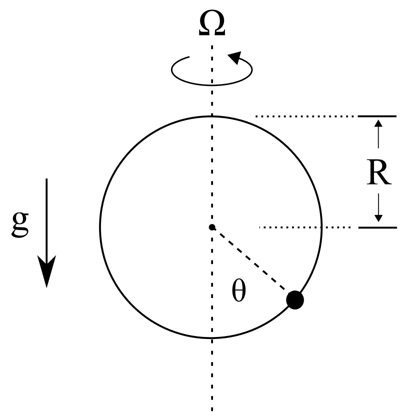
[1] f is a nice and smooth function.



Hint: This is a bit of Newtonian mechanics again. The bead is affected by the gravitational force and the normal force. In order to stay on the hoop, the bead must experience a centripetal acceleration. This acceleration fixes the radial component of the total force acting on the bead.

(3 points)

3 Bead on a rotating hoop



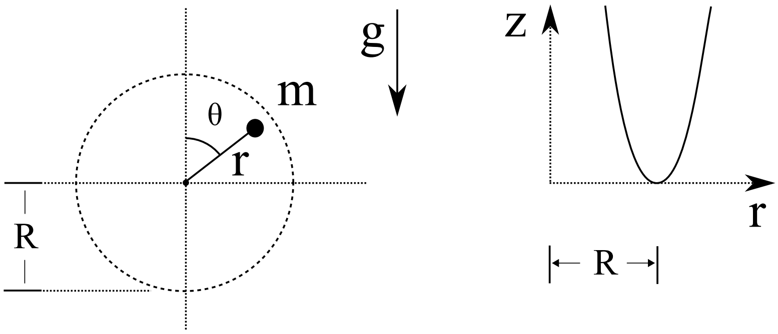
This exercise illustrates that the Lagrangian method also works for systems where the constraints depend on time. We consider a bead of mass m, which slides without friction on a hoop of radius R that rotates around its vertical axis with a constant angular speed Ω. Hence, the bead is constrained to only move on the hoop, but the hoop does in turn rotate. The bead is affected by gravity.

a) *Derive the Lagrangian expressed in terms of the angle θ that the bead makes with the vertical downwards direction.* **(3 points)**

b) *Find the equation of motion of the bead.* **(2 points)**

4 Gold star exercise: Obtaining constraint forces

This problem gives no points! However, if you are interested in how the idealized constraint forces come about, this is the exercise for you.



In exercise 2 we took for granted that the bead stays at the fixed radius R of the hoop. In other words, we assume that there are constraint forces that keep the bead precisely at the right radius. One can realize that this is an idealization. The restoring force comes from the hoop pushing the bead back, when the bead deviates slightly from the correct radius. Here we will analyze this problem by replacing the perfect constraint with a potential. We will see that one can regain the ideal constraint for very steep potentials.

Precisely as in problem 2 we consider a bead on a hoop, but we additionally allow the bead to change its radial coordinate r. Hence, we now have two coordinates θ and r. However, we also include a radial potential (in addition to the gravitational potential) $V(r) = \frac{1}{2\epsilon}(r-R)^2$, where $\epsilon > 0$. Hence, this is a quadratic potential with minimum at $r = R$, and this potential becomes more and more steep the smaller ϵ is.

The solutions θ and r to the Euler-Lagrange equations are functions of time, but also of the parameter ϵ. Intuitively, is seems likely that if ϵ is very small (and thus the potential very steep), then $r(\epsilon, t) \approx R$, and we would regain the ideal case in exercise 2. In order to analyze this we are going to apply a very common type of trick, namely a perturbation expansion.

a) *Derive the Lagrangian of this system, and show that the Euler-Lagrange equations are*

$$\begin{aligned} &mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgr\sin\theta = 0, \\ &m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta + \frac{1}{\epsilon}(r-R) = 0. \end{aligned} \tag{2}$$

(0 points)

b) Next we expand r and θ in power-series around $\epsilon = 0$,

$$\begin{aligned} \theta(\epsilon, t) =& \theta_0(t) + \epsilon\theta_1(t) + \epsilon^2\theta_2(t) + \cdots, \\ r(\epsilon, t) =& r_0(t) + \epsilon r_1(t) + \epsilon^2 r_2(t) + \cdots. \end{aligned}$$

Insert the above expansions into the equations (2). For each equation, collect terms of equal order in ϵ. We only need the orders $\frac{1}{\epsilon}$ and 1, so you can ignore ϵ, ϵ^2, ϵ^3, etc.

Hint: You will get one equation from the $\frac{1}{\epsilon}$-order, and two equations from the 1-order.

(0 points)

c) Simplify the equations that you obtained in b). *How does the result relate to the Euler-Lagrange equation in problem 3 a)?* The radial potential yields the force $F(t) = -V'\big(r(t)\big)$. *For the approximation $r(t) \approx r_0(t) + \epsilon r_1(t)$, determine the force, and compare with (1).* **(0 points)**

Remark: Here we only consider the lowest orders of perturbation. To estimate how the bead deviates from the ideal evolution of the perfect constraints, one can include higher orders in the expansion (such as θ_1 and r_2). Unfortunately, the resulting expressions are not very pleasant.

Comment: Apart from analyzing how constraint forces comes about, this exercise also introduces the notion of perturbation theory, which is something that you will encounter many times in other parts of physics.