Classical Mechanics

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Exercise sheet 9 Due: Thursday December 16 at 24:00

1 Particle in an electromagnetic field

For a particle with mass m, charge e, and position $\vec{r}$, which moves in an electromagnetic field, the Lagrangian can be written as

$$L(\vec{r},\dot{\vec{r}}) = \frac{m}{2}\dot{\vec{r}}^2 - e\phi(\vec{r},t) + e\vec{A}(\vec{r},t)\cdot\dot{\vec{r}}, \tag{1}$$

where $\phi(\vec{r},t)$ is a real-valued function (the scalar potential), and $\vec{A}(\vec{r},t)$ is a vector-valued function (the vector-potential)

Please do not panic if you are not familiar with electromagnetism and vector-potentials, think of (1) as simply being yet another (maybe strange looking) Lagrangian of a particle.

a) Introduce the Cartesian coordinates $\vec{r} = (r_1, r_2, r_3)$ and $\vec{A} = (A_1, A_2, A_3)$. *Show that the Euler-Lagrange equations obtained from the Lagrangian* (1) *can be written as*

$$m\ddot{r}_j + e\left(\frac{\partial\phi}{\partial r_j} + \frac{\partial A_j}{\partial t}\right) - e\sum_{k=1}^{3}\dot{r}_k\left(\frac{\partial A_k}{\partial r_j} - \frac{\partial A_j}{\partial r_k}\right) = 0, \quad j = 1,2,3. \tag{2}$$

Hint: It can be useful to first rewrite (1) in terms of the Cartesian components

$$L(\vec{r},\dot{\vec{r}}) = \frac{m}{2}\sum_{k=1}^{3}\dot{r}_k^2 - e\phi(\vec{r},t) + e\sum_{k=1}^{3}A_k(\vec{r},t)\dot{r}_k.$$

(2 points)

b) The electric field can be obtained as $\vec{E}(\vec{r},t) = -\nabla\phi - \frac{\partial}{\partial t}\vec{A}$ and the magnetic field as $\vec{B}(\vec{r},t) = \nabla\times\vec{A}$. *Use these relations in order to rewrite* (2) *in terms of the electric and magnetic fields, instead of* ϕ *and* $\vec{A}$.

Hint: What do you get if you expand $\dot{\vec{r}}\times(\nabla\times\vec{A})$?

(3 points)

c) Let $\chi(\vec{r},t)$ be a real-valued function (a scalar function), and suppose that we change ϕ and $\vec{A}$ into the new functions ϕ' and $\vec{A}'$ by

$$\phi' = \phi - \frac{\partial\chi}{\partial t}, \quad \vec{A}' = \vec{A} + \nabla\chi.$$

This is called a gauge-transformation. *Show that this leaves* $\vec{E}$ *and* $\vec{B}$ *invariant.*

(2 points)

d) In the following, we shall consider an alternative way to obtain the gauge-invariance, by using Lagrangians. As a first step, let L and L' be two Lagrangians that are related as

$$L'(\vec{r},\dot{\vec{r}},t) = L(\vec{r},\dot{\vec{r}},t) + \frac{d}{dt}f(\vec{r},t),$$

for some real-valued function that does not depend on $\dot{\vec{r}}$. *Show that*

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{r}_j} - \frac{\partial L'}{\partial r_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}_j} - \frac{\partial L}{\partial r_j}.$$

From this it follows that the Euler-Lagrange equations corresponding to L' and L are identical.

Hint: The main challenge is to keep track of partial and total derivatives. Note that it is important that f does not depend on $\dot{\vec{r}}$. Also, assume that f is smooth and well behaved enough that we can swap the order of derivatives.

(3 points)

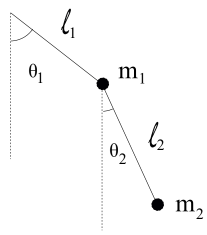
e) As an application of the result in d), let L' be the new Lagrangian that is obtained if we substitute ϕ and $\vec{A}$ in (1) by ϕ' and $\vec{A}'$. *Show that L and L' only differ by a total time-derivative of some function of $\vec{r}$ and t.* In view of d) the conclusion is that the resulting equations of motion remain the same after a gauge-transformation.

Remark: Note that you can solve this problem even if you did not manage to solve d).

(2 points)

2 Double pendulum without gravity

Consider a double-pendulum where the two masses move in a plane. Mass m_1 is attached to a fixed point via a massless rod of length ℓ_1. A second mass m_2 is in turn attached by a massless rod of length ℓ_2 to mass m_1. In this problem we assume that the masses are *not* affected by gravity.



a) *Show that the Lagrangian with respect to the angles θ_1 and θ_2 as in the figure can be written as*

$$L = A\dot{\theta}_1^2 + B\dot{\theta}_2^2 + C\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1),$$

and determine the constants A, B and C. Note that the since there is no gravity, there is no potential energy. **(3 points)**

b) Recall from the lecture that a transformation $\vec{\Phi}^{(s)} : \mathbb{R}^n \to \mathbb{R}^n$ with $\vec{\Phi}^{(0)}(\vec{q}) = \vec{q}$ is a symmetry transformation of a Lagrangian $L(\vec{q}, \dot{\vec{q}})$ if

$$\frac{d}{ds}\Big|_{s=0} L\Big(\vec{\Phi}^{(s)}(\vec{q}(t)), \frac{d}{dt}\vec{\Phi}^{(s)}(\vec{q}(t))\Big) = 0.$$

Show that the transformation

$$(\theta_1, \theta_2) \mapsto \vec{\Phi}^{(s)}(\theta_1, \theta_2) = (\theta_1 + s, \theta_2 + s).$$

is a symmetry of the Lagrangian in a). **(2 points)**

c) *Derive the conserved quantity from Noether's theorem, with respect to the transformations $\vec{\Phi}^{(s)}$ in b).*

(3 points)