Classical Mechanics

WS 2021/22 — Test Exam

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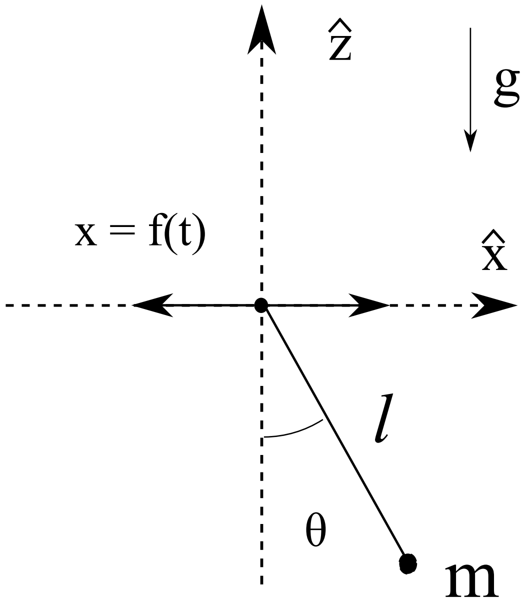
1 Obtaining the Lagrange function



A massless rod of length l is attached to a point that moves in the horizontal direction by the function $x = f(t)$. At the other end of the rod is attached a point mass m. The rod swings in the z, x-plane, where we let $\hat{z}$ be the unit vector in the vertical direction upwards, and $\hat{x}$ is the unit vector in the horizontal direction to the right. The mass m is affected by a constant gravitational acceleration g in the negative z-direction.

Determine the Lagrange function for this system in terms of the angle θ between the rod and the vertical (downwards) direction (see figure).

2 Euler-Lagrange equations

A particle is described by the following Lagrange function

$$L(r, \theta, \dot{r}, \dot{\theta}) = \frac{m_0}{2} r^3 \dot{r}^2 + \gamma \cos(\alpha r) \dot{\theta}^2,$$

where r is the radial coordinate, and θ the angular coordinate of the particle in a plane, and where m_0, α and γ are constant.

a) *Obtain the equations of motion for r and θ via the Euler-Lagrange equations.*

b) *Determine the cyclic coordinates and the corresponding conserved quantity.*

c) *Use the result from b) in order to obtain an equation of motion for r, which does not depend on θ.*

Hint: Eliminate the cyclic coordinate from one of the equations of motion.

3 From Lagrange to Hamilton

Consider the following Lagrange function

$$L(x, \dot{x}, z, \dot{z}) = \frac{m}{2}\dot{x}^2 + \beta x\dot{z} + \gamma\dot{z}^2$$

where $m > 0$, $\beta > 0$ and $\gamma > 0$ are constants.

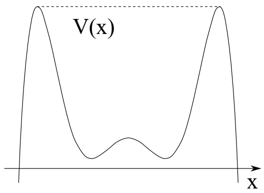
a) *Determine the conjugate momenta to x and z.*

b) *Obtain the Hamilton function.*

c) *Determine Hamilton's equations of motion.*

4 Phase space portrait

Consider a particle with mass m that moves in one dimension under the influence of a potential $V(x)$ of the form in the figure.



a) *Which are the equilibrium points? What can you say about their stability?*

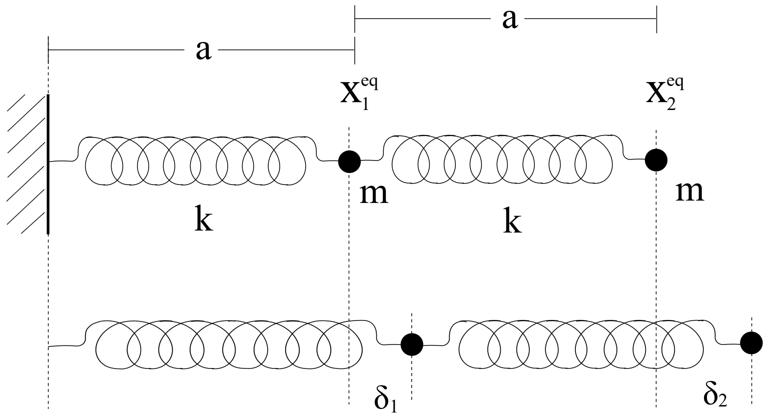
b) *Sketch the phase space portrait. Indicate the elliptic and hyperbolic fixpoints. Draw the energy level curves that are on the same energy as hyperbolic fixpoints (the separatrices). Indicate the direction of the flow in the different regions separated by the separatrices.* Note that you only need to make a qualitative sketch.

Remarks: Note that the potential is symmetric, such the largest maxima have the same energy. Consider what effect this has on the shape of the phase space flow.

5 Normal modes

Two particles of equal mass m are confined to move on a line ($\mathbb{R}$). Particle 1 is attached to a wall via a harmonic force with a spring constant $k > 0$. Particle 1 also interacts with particle 2 via a harmonic force with spring constant k. If x_1 is the distance of particle 1 from the wall, and x_2 the distance of particle 2 from the wall, then the system is in mechanical equilibrium when the positions of the atoms are such that $x_1 = x_1^{\text{eq}} = a$, $x_2 = x_2^{\text{eq}} = 2a$, for some constant $a > 0$. In terms of the *deviations* from equilibrium $\delta_1 = x_1 - x_1^{\text{eq}} = x_1 - a$ and $\delta_2 = x_2 - x_2^{\text{eq}} = x_2 - 2a$, the equations of motion can be written

$$\begin{aligned} m\ddot{\delta}_1 &= -2k\delta_1 + k\delta_2 \\ m\ddot{\delta}_2 &= k\delta_1 - k\delta_2. \end{aligned}$$



a) The equations of motion can be rewritten

$$\frac{d^2}{dt^2}\vec{\delta} = \boldsymbol{M}\vec{\delta}, \quad \vec{\delta} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix},$$

for some matrix $\boldsymbol{M}$. *Determine the matrix* $\boldsymbol{M}$.

b) In order to find the normal modes, make the ansatz $\vec{\delta}(t) = e^{i\omega t}\vec{v}$, where $\vec{v}$ is a time-independent vector, and ω a real number. *Show that this leads to an eigenvalue problem of the form*

$$\lambda\vec{v} = \boldsymbol{M}\vec{v},$$

where the eigenvalue λ *is a function of* ω. *Determine this function.*

c) *Determine the eigenvalues* λ *of* $\boldsymbol{M}$.

d) *For each eigenvalue* λ*, make a qualitative sketch of the motion of the the two atoms of the corresponding normal mode.* You only need to indicate the relative direction of the motion of the atoms; whether they move in opposite direction of each other, or in the same direction.

6 Conserved quantities via Poisson brackets

Consider a system with coordinates Q_1, Q_2 and corresponding conjugate momenta P_1, P_2 with Hamilton function

$$H = e^{-2Q_1} + P_1^2 e^{2Q_1} + e^{-2P_2} e^{2Q_1} + 2P_1 e^{-P_2} e^{2Q_1} + Q_2^2 e^{2P_2} + 2Q_1 Q_2 e^{P_2} + Q_1^2. \tag{1}$$

Consider the function

$$W = \alpha Q_2 e^{P_2} + \beta Q_1.$$

Use Poisson brackets to determine for which values of $\alpha, \beta \in \mathbb{R}$ *the quantity* W *is a conserved quantity with respect to* H *in* (1).

7 Noether's theorem

Consider a Lagrangian of the following form

$$L(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3) = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 + \frac{m_3}{2}\dot{x}_3^2 + \alpha\dot{x}_3(x_2 - x_1) + \beta\dot{x}_2(x_1 - x_3) + \gamma\dot{x}_1(x_3 - x_2),$$

where m_1, m_2, m_3 are masses, and $\alpha, \beta, \gamma \neq 0$ are constants.

a) *Show that*

$$\frac{d}{ds}\Big|_{s=0} L\left(\vec{\Phi}^{(s)}\left(\vec{q}(t)\right), \frac{d}{dt}\vec{\Phi}^{(s)}\left(\vec{q}(t)\right)\right) = 0,$$

for transformations on the form

$$(x_1, x_2, x_3) \mapsto \vec{\Phi}^{(s)}(x_1, x_2, x_3) = (x_1 + s, x_2 + s, x_3 + s).$$

b) *Obtain the conserved quantity from Noether's theorem, with respect to the transformation in a).*