Quantum Mechanics

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WS 24/25

Sheet 1: Bonus sheet **Saturday October 12 at 24:00 Uhr**

The Fourier transform of a function $\psi : \mathbb{R} \to \mathbb{C}$ is defined as

$$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} \psi(x),$$

with the inverse transform

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} \psi(k).$$

The second relation says that we can interpret every function ψ as a superposition of various waves e^{ikx}, where each comes with an amplitude $\psi(k)$. The Fourier transform is an important tool for the analysis of wave equations, in both classical and quantum mechanics. In quantum mechanics, it is closely connected to the notion of momentum.

1 Fourier transforms and operations on functions

Remark: If you have followed the lecture "Mathematik für Studierende der Physik II' last term, then you will recognise these exercises.

In this exercise, we will investigate what happens to the Fourier transform, when the function is complex conjugated, shifted, differentiated or squeezed. All the tasks can be solved using standard rules for the calculation of integrals.

a) *Conjugation* $\sim$ *Conjugation and reflection about origin*: For a complex number z, we let z^* denote its complex conjugate. *Show that*

$$\textit{if} \quad \phi(x) = \psi(x)^*, \quad \textit{then} \quad \tilde{\phi}(k) = \tilde{\psi}(-k)^*.$$

Remark: In particular, whenever $\tilde{\psi}(-k) = \tilde{\psi}(k)^*$, it follows that ψ is real-valued.

(2 points)

b) *Translation* $\sim$ *Multiplication with a complex phase factor*: Let $a \in \mathbb{R}$. *Show that*

$$\textit{if} \quad \phi(x) = \psi(x-a), \quad \textit{then} \quad \tilde{\phi}(k) = e^{-ika}\tilde{\psi}(k).$$

Also, show that

$$\textit{if} \quad \phi(x) = e^{iax}\psi(x), \quad \textit{then} \quad \tilde{\phi}(k) = \tilde{\psi}(k-a).$$

(4 points)

c) *Differentiation* $\sim$ *Multiplikation with the argument*: Assume that ψ approaches zero at inifinity, i.e.,

$$\lim_{x\to\pm\infty} \psi(x) = 0.$$

Show that

$$\textit{if} \quad \phi(x) = \psi'(x), \quad \textit{then} \quad \tilde{\phi}(k) = ik\tilde{\psi}(k).$$

Hint: Use partial integration.

(3 points)

d) *Stretching* $\sim$ *Compression*: Let $a \in \mathbb{R}$ with $a \neq 0$. *Show that*

$$\text{if} \quad \phi(x) = \psi(x/a), \quad \text{then} \quad \tilde{\phi}(k) = a\tilde{\psi}(ak).$$

(2 points)

2 Fourier transform of a wave-packet

The purpose of this exercise is to determine the Fourier transformation of a Gaussian wave-packet of the form

$$\psi(x) = A\sin(k_0 x)e^{-x^2/a^2}. \tag{1}$$

For this task you can use the Gauss-integral

$$\int_{-\infty}^{\infty} dx\, e^{-(x+\beta)^2} = \sqrt{\pi},$$

which holds for every $\beta \in \mathbb{C}$.

a) We start with a somewhat simpler special case of what we want to show. Let $\psi(x) = e^{-x^2}$. *Show that*

$$\tilde{\psi}(k) = \frac{1}{\sqrt{2}} e^{-k^2/4}.$$

Hint: Use quadratic completion in order to put the integrand in a form such that you can make use of the above Gauss integral.

(3 points)

b) Let $\psi(x) = e^{-x^2/a^2}$. *Show that*

$$\tilde{\psi}(k) = \frac{a}{\sqrt{2}} e^{-\frac{a^2k^2}{4}}.$$

Hint: Combine the results in a) with your findings in problem 1.

(2 points)

c) Next, assume instead that $\psi(x) = e^{ik_0x}e^{-x^2/a^2}$. *Show that*

$$\tilde{\psi}(k) = \frac{a}{\sqrt{2}} e^{-\left(\frac{a}{2}(k-k_0)\right)^2}.$$

(2 points)

d) Finally, let $\psi(x)$ be as in (1). *Show that*

$$\psi(k) = -i\frac{aA}{\sqrt{8}}\left(e^{-\left(\frac{a}{2}(k-k_0)\right)^2} - e^{-\left(\frac{a}{2}(k+k_0)\right)^2}\right).$$

HInt: Write the sine function as a superposition of two exponential functions, and use the results in c).

(2 points)