Quantum Mechanics

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WS 24/25

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1 Properties of the angular momentum operators

A triple of operators (J_1, J_2, J_3) can be interpreted as representing the angular momentum of a physical system, if they satisfy the angular momentum commutation relations

$$[J_1, J_2] = i\hbar J_3, \quad \text{and cyclic permutations,} \tag{1}$$

or more compactly

$$[J_k, J_l] = i\hbar \sum_n \epsilon_{n,k,l} J_n,$$

where ϵ is the Levi-Civita symbol. In order to work with angular momentum, it is useful to introduce some additional operators. In particular the absolute value square of the angular momentum

$$J^2 = J_1^2 + J_2^2 + J_3^2,$$

which in some sense corresponds to magnitude. We also have the ladder operators[1]

$$J_\pm = J_1 \pm iJ_2.$$

a) *Show that*

$$[J^2, J_3] = 0. \tag{2}$$

(2 points)

Remark: Recall that Hermitian operators that commute share a common eigenbasis. This eigenbasis is very convenient for analysing angular momentum. Each such joint egienstate is characterised by two quantum numbers; the eigenvalue of J^2 and the eigenvalue of J_3. We will see more of this in exercise 2.

b) *Show that*

$$[J_3, J_\pm] = \pm\hbar J_\pm.$$

(2 points)

c) *Show that*

$$J_- J_+ = J^2 - J_3^2 - \hbar J_3.$$

(2 points)

[1]One may note the analogy with the ladder operators $a = \frac{1}{\sqrt{2}}(\tilde{X} + i\tilde{P})$ and $a^\dagger = \frac{1}{\sqrt{2}}(\tilde{X} - i\tilde{P})$, for the Harmonic oscillator.

2 Spin 1

As mentioned above, the commutation relation (2) means that J^2 and J_3 share a common eigenbasis, which we denote by $|j, m\rangle$. Here, j can either be an integer or a half-odd-integer (i.e. $0, 1, 2, \ldots$ or $1/2, 3/2, 5/2, \ldots$) and

$$\begin{aligned} J^2|j,m\rangle &= j(j+1)\hbar^2|j,m\rangle, \\ J_3|j,m\rangle &= m\hbar|j,m\rangle, \quad m = -j, -j+1, \cdots, j-1, j. \end{aligned}$$

Hence, for each fixed j there are $2j+1$ possible values of m. By the ladder operators $J_\pm$ we can step between the values of the quantum number m, in the sense that

$$J_+|j,m\rangle = \hbar\sqrt{(j-m)(j+m+1)}|j,m+1\rangle, \quad J_-|j,m\rangle = \hbar\sqrt{(j+m)(j-m+1)}|j,m-1\rangle.$$

Here "spin 1" simply means $j = 1$, in which case the corresponding set of joint eigenvectors are $|1,-1\rangle, |1,0\rangle, |1,1\rangle$ that span a 3-dimensional space. Here we wish to represent the operators $J^2, J_1, J_2, J_3, J_\pm$ as matrices in this basis.

a) *First of all, determine the matrix*

$$\boldsymbol{M}(J^2) = \begin{bmatrix} \langle 1,1|J^2|1,1\rangle & \langle 1,1|J^2|1,0\rangle & \langle 1,1|J^2|1,-1\rangle \\ \langle 1,0|J^2|1,1\rangle & \langle 1,0|J^2|1,0\rangle & \langle 1,0|J^2|1,-1\rangle \\ \langle 1,-1|J^2|1,1\rangle & \langle 1,-1|J^2|1,0\rangle & \langle 1,-1|J^2|1,-1\rangle \end{bmatrix}$$

and analogously for $\boldsymbol{M}(J_3)$.

(2 points)

b) *Determine* $\boldsymbol{M}(J_\pm)$.

Hint: Think about which matrix elements have to be zero.

(2 points)

c) *Finally determine* $\boldsymbol{M}(J_1)$ *and* $\boldsymbol{M}(J_2)$.

Hint: There was a reason for why we determined $\boldsymbol{M}(J_\pm)$ *before* we determine $\boldsymbol{M}(J_1)$ and $\boldsymbol{M}(J_2)$.

(2 points)

d) What, wait?! In the lecture we derived the generators for the spin-1 case and got[2]

$$\boldsymbol{l}_1 = i\hbar\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{l}_2 = i\hbar\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{l}_3 = i\hbar\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This does *not* coincide with the matrices $\boldsymbol{M}(J_1), \boldsymbol{M}(J_2), \boldsymbol{M}(J_3)$ that we get above. Is something wrong? No, nothing is wrong, we have just used two different bases for the representing J_1, J_2, J_3 on the spin-1 subspace, where we in the lecture used a Cartesian coordinate system in

[2] Well, strictly speaking, we got $l_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $l_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, $l_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. However, this is a more trivial difference in terms of how choose generators, where $U_{\omega_1,\omega_2,\omega_3}(\chi) = e^{-i\chi(\omega_1\boldsymbol{l}_1+\omega_2\boldsymbol{l}_2+\omega_3\boldsymbol{l}_3)/\hbar} = e^{\chi(\omega_1 l_1+\omega_2 l_2+\omega_3 l_3)}$. The choice $\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}_3$ corresponds to the preference in quantum mechanics for Hermitian generators (and $\hbar$ relates to the wish to keep track of units).

$\mathbb{R}^3$. However, the eigenbasis $|1,1\rangle, |1,0\rangle, |1,-1\rangle$ corresponds to a complex unitary transformation of those Cartesian coordinates.[3] Define the new ON-basis

$$|e_1>= -\frac{1}{\sqrt{2}}|1,1\rangle + \frac{1}{\sqrt{2}}|1,-1>, \quad |e_2>= \frac{i}{\sqrt{2}}|1,1> + \frac{i}{\sqrt{2}}|1,-1\rangle, \quad |e_3\rangle = |1,0\rangle$$

Show that

$$\begin{bmatrix} \langle e_1|J_3|e_1\rangle & \langle e_1|J_3|e_2\rangle & \langle e_1|J_3|e_3\rangle \\ \langle e_2|J_3|e_1\rangle & \langle e_2|J_3|e_2\rangle & \langle e_2|J_3|e_3\rangle \\ \langle e_3|J_3|e_1\rangle & \langle e_3|J_3|e_2\rangle & \langle e_3|J_3|e_3\rangle \end{bmatrix} = \boldsymbol{l}_3. \tag{3}$$

Hint: It can be a good idea to express J_3 in terms of some outer product of bras and kets.

Remark: One can analogously obtain $\boldsymbol{l}_1$ as the matrix representation of J_1 in the $\{|e_k\rangle\}_{k=1}^3$ basis, as well as $\boldsymbol{l}_2$ as the matrix representation of J_2. However, it is tiresome enough to show (3).

(2 points)

3 Orbital angular momentum

Recall that for a classical particle with position $\boldsymbol{x} \in \mathbb{R}^3$ and momentum $\boldsymbol{p}$, the angular momentum (with respect to the origin) is given by $\boldsymbol{x} \times \boldsymbol{p}$. With this classical expression in mind, we could attempt to define the quantum orbital angular momentum operators $\boldsymbol{L} = (L_1, L_2, L_3)$ as

$$\boldsymbol{L} = \boldsymbol{X} \times \boldsymbol{P}, \quad \text{or equivalently} \quad L_j = \sum_{m,n} \epsilon_{j,m,n} X_m P_n,$$

where $\boldsymbol{X} = (X_1, X_2, X_3)$ and $\boldsymbol{P} = (P_1, P_2, P_3)$ are the position and momentum operators, respectively, with respect to a Cartesian coordinate system.

a) As mentioned in exercise 1, it is legitimate to regard $\boldsymbol{L} = (L_1, L_2, L_3)$ as angular momentum operators, if they satisfy the angular momentum commutation relations. *Show that the $\boldsymbol{L} = (L_1, L_2, L_3)$ satisfy the angular momentum commutation relations (1).*

Hint: One can directly evaluate $[L_1, L_2]$, $[L_2, L_3]$ and $[L_3, L_1]$, with $L_1 = X_2P_3 - X_3P_2$, $L_2 = X_3P_1 - X_1P_3$, $L_3 = X_1P_2 - X_2P_1$, or one can do all of them in one single go by using the formulations in terms of the Levi-Civita symbol. In case you go for the Levi-Civita symbols, then you can make use of the relation $\sum_i \epsilon_{iul}\epsilon_{ist} = \delta_{us}\delta_{lt} - \delta_{ut}\delta_{ls}$.

(2 points)

b) From the lecture we know that the angular momentum operators are the generators of rotations. Nevertheless, in the following we explicitly confirm this for the orbital angular momentum. For the sake of simplicity[4] we focus on rotations around the 3-axis (the z-axis). The component L_3, represented as a differential operator in Cartesian coordinates, is

$$\langle \boldsymbol{x}|L_3|\psi\rangle = \langle \boldsymbol{x}|(X_1P_2 - X_2P_1)|\psi\rangle = -i\hbar x_1 \frac{\partial}{\partial x_2}\psi(\boldsymbol{x}) + i\hbar x_2 \frac{\partial}{\partial x_1}\psi(\boldsymbol{x}).$$

Show that for spherical coordinates, L_3 takes the form

$$\langle \theta, \phi, r|L_3|\psi\rangle = -i\hbar \frac{\partial}{\partial \phi}\psi.$$

[3] We can of course not do such a transformation in $\mathbb{R}^3$. Here, we make use of the fact that we operate in $\mathbb{C}^3$.

[4] The components L_1 and L_2 get a bit tiresome in the standard spherical coordinate system, and the calculations in c) would get even more horrible.

Recall that for the spherical coordinates we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r \sin\theta \cos\phi \\ r \sin\theta \sin\phi \\ r\cos\theta \end{bmatrix}.$$

(2 points)

c) Suppose that we have a wave-function $\psi(\theta,\phi,r)$ in terms of the spherical coordinate system. A rotation $R_3(\chi)$ by an angle χ about the 3-axis corresponds to $[R_3(\chi)\psi](\theta,\phi,r) = \psi(\theta,\phi-\chi,r)$. The generator L_3 corresponds to the unitary operation $e^{-i\chi L_3/\hbar}$, which in the spherical coordinate system takes the shape $\langle\theta,\phi,r|e^{-i\chi L_3/\hbar}|\psi\rangle = \left(e^{-\chi\frac{\partial}{\partial\phi}}\psi\right)(\theta,\phi,r)$. *Show that*

$$\left(e^{-\chi\frac{\partial}{\partial\phi}}\psi\right)(\theta,\phi,r) = \psi(\theta,\psi-\chi,r).$$

In other words, you should confirm that the L_3 component of the orbital angular momentum indeed is the generator of rotations around the 3-axis.

Hint: As for so many other occasions, we have a friend named Taylor.

(2 points)