Quantum Mechanics

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1 Addition of angular momenta

We have seen that for an angular momentum[1] $\boldsymbol{J} = (J_x, J_y, J_z)$ we can determine a joint eigenbasis $|j, m\rangle$ of the pair of operators $\boldsymbol{J}^2, J_z$, where

$$\boldsymbol{J}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle, \quad J_z|j,m\rangle = m\hbar|j,m\rangle.$$

In this exercise we consider the question what happens if we combine *two* physical systems that each carry an angular momentum. To be more precise, we have two angular momenta $\boldsymbol{J}_1 = (J_{1x}, J_{1y}, J_{1z})$ and $\boldsymbol{J}_2 = (J_{2x}, J_{2y}, J_{2z})$. Here, both $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ independently satisfy the angular momentum commutation relations, but all the angular momentum operators of one system commutes with all angular momentum operators of the other[2] (so $[\boldsymbol{J}_1, \boldsymbol{J}_2] = 0$, meaning that $[J_{1x}, J_{2x}] = 0$, $[J_{1x}, J_{2y}] = 0$ etc). A consequence of the latter is that both of $\boldsymbol{J}_1^2, J_{1z}$ commute with both of $\boldsymbol{J}_2^2, J_{2z}$. In other words, the four operators $\boldsymbol{J}_1^2, \boldsymbol{J}_2^2, J_{1z}, J_{2z}$ form a mutually commuting set, and have the simultaneous eigenbasis[3]

$$|j_1, j_2, m_1, m_2\rangle = |j_1, m_1\rangle|j_2, m_2\rangle.$$

meaning that

$$\begin{aligned} \boldsymbol{J}_1^2|j_1,j_2,m_1,m_2\rangle =& j_1(j_1+1)\hbar^2|j_1,j_2,m_1,m_2\rangle, \quad \boldsymbol{J}_2^2|j_1,j_2,m_1,m_2\rangle = j_2(j_2+1)\hbar^2|j_1,j_2,m_1,m_2\rangle, \\ J_{1z}|j_1,j_2,m_1,m_2\rangle =& m_1|j_1,j_2,m_1,m_2\rangle, \quad J_{2z}|j_1,j_2,m_1,m_2\rangle = m_2|j_1,j_2,m_1,m_2\rangle. \end{aligned}$$

This is of course an orthonormal basis as good as any other. However, for analysing the physics of the combined system, it is often more useful to consider another orthonormal basis, related to the *total* angular momentum $\boldsymbol{J} = (J_x, J_y, J_z)$ defined by

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2 \quad \text{or in components} \quad J_x = J_{1x} + J_{2x}, \quad J_y = J_{1y} + J_{2y}, \quad J_z = J_{1z} + J_{2z}. \tag{1}$$

As we have seen in the lecture, $\boldsymbol{J}$ in (1) is indeeed is an angular momentum operator, i.e., its components satisfy the angular momentum commutation relations.

a) Before we start, just a general observation (which I would better have included on the last sheet). *Show that if* $\boldsymbol{L}$ *is an angular mometum, then* $[\boldsymbol{L}^2, L_\pm] = 0$. The reason for why I use $\boldsymbol{L}$ here, is in order to emphasisze that this holds for $\boldsymbol{J}$, as well as for $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$.

Hint: Note that not only $[\boldsymbol{L}^2, L_z] = 0$, but also $[\boldsymbol{L}^2, L_x] = 0$ and $[\boldsymbol{L}^2, L_y] = 0$.

(2 points)

[1] Since we here deal with systems 1 and 2, I believe that it is less confusing to denote the components by x, y, z, rather than $1, 2, 3$.

[2] It is generally the the case that operators on one subsystem commutes with all operators on another subsystem. More precisely, if A_1 is and operator on $\mathcal{H}_1$ and B_2 is an operator on $\mathcal{H}_2$, then $[A_1 \otimes \hat{1}_2, \hat{1}_1 \otimes B_2] = 0$. Often, one would simplify the notion and only write $[A_1, B_2] = 0$.

[3] To be a more formally picky, we should rather write the combined eigenbasis with the tensor product symbol $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$, since the combined basis elements are members of the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, i.e., the tensor product of Hilbert space $\mathcal{H}_1$ of system 1 and Hilbert space $\mathcal{H}_2$ of system 2. Similarly, the combined momentum J in (1) should strictly speaking be written $\boldsymbol{J} = \boldsymbol{J}_1 \otimes \hat{1}_2 + \hat{1}_1 \otimes \boldsymbol{J}_2$, meaning $J_x = J_{1x} \otimes \hat{1}_2 + \hat{1}_1 \otimes J_{2x}$ etc. However, this all result in very extensive and messy expressions, so one often tends to drop these things.

b) *Show that*

$$\boldsymbol{J}^2 = \boldsymbol{J}_1^2 + \boldsymbol{J}_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}, \tag{2}$$

where $J_{1\pm}$ *and* $J_{2\pm}$ *are the ladder operators for systems* 1 *and* 2*.*

Hint: One important step is to express $J_{1x}J_{2x} + J_{1y}J_{2y}$ in terms of $J_{1\pm}$ and $J_{2\pm}$.

(4 points)

c) From $\boldsymbol{J}$ being an angular momentum, it follows directly that $\boldsymbol{J}^2$ commutes with J_z. *Show that* $\boldsymbol{J}_1^2$ *and* $\boldsymbol{J}_2^2$ *commute with both* $\boldsymbol{J}^2$ *and* J_z*.*

Hint: There was a reason for why we did **b)**. Here also comes the reason for why I squeezed in **a)**.

(2 points)

d) *Show that* $\boldsymbol{J}^2$ *does not commute with* J_{1z}*, nor with* J_{2z}*.*

(2 points)

e) From **c)** we know that $\boldsymbol{J}_1^2$, $\boldsymbol{J}_2^2$, $\boldsymbol{J}^2$, $\boldsymbol{J}_z$ form a jointly commuting set. Hence, there exists a joint eigenbasis $|j_1, j_2, j, m\rangle$, such that

$$\boldsymbol{J}_1^2|j_1, j_2, j, m\rangle = j_1(j_1+1)\hbar^2|j_1, j_2, j, m\rangle, \quad \boldsymbol{J}_2^2|j_1, j_2, j, m\rangle = j_2(j_2+1)\hbar^2|j_1, j_2, j, m\rangle,$$
$$\boldsymbol{J}^2|j_1, j_2, j, m\rangle = j(j+1)\hbar^2|j_1, j_2, j, m\rangle, \quad J_z|j_1, j_2, j, m\rangle = m|j_1, j_2, j, m\rangle.$$

From **d)** we can (unfortunately[4]) also conclude that the basis $|j_1, j_2, j, m\rangle$ does *not* coincide with $|j_1, j_2, m_1, m_2\rangle$. We can express one basis in terms of the other

$$|j_1, j_2, j, m\rangle = \sum_{m_1, m_2} |j_1, j_2, m_1, m_2\rangle\langle j_1, j_2, m_1, m_2|j_1, j_2, j, m\rangle,$$

where $\langle j_1, j_2, m_1, m_2|j_1, j_2, j, m\rangle$ are referred to as the Clebsch-Gordan coefficients. A good thing about the Clebsch-Gordan coefficients is that many of them actually are zero. *Show that if* $m \neq m_1 + m_2$*, then*

$$\langle j_1, j_2, m_1, m_2|j_1, j_2, j, m\rangle = 0.$$

Hint: Note that $J_z - J_{1z} - J_{2z} = 0$. Take this as a starting point.

(2 points)

2 Coupling of two spin-half systems

Here we take a closer look at the special case of two spin-half particles. As we know from the previous exercise, we have the joint orthonormal eigenbasis $|j_1, j_2, m_1, m_2\rangle$. However, since $j_1 = j_2 = \frac{1}{2}$ throughout this exercise, we will simplify the notion and only write $|m_1, m_2\rangle$ for these basis elements, where we note that the possible values of m_1 and m_2 are $(m_1, m_2) = (-\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$. Hence, the joint Hilbert space of the two spin-half is four-dimensional.

Similarly, we will only write $|j, m\rangle$ for the elements $|j_1, j_2, j, m\rangle$ in the joint orthonormal eigenbasis of $\boldsymbol{J}_1^2$, $\boldsymbol{J}_2^2$, $\boldsymbol{J}^2$, $\boldsymbol{J}_z$.

[4] It is indeed unfortunate, the whole business of angular momentum addition in quantum mechanics would have been so much simpler if these eigenbases would have coincided.

a) The possible values of j are $j = |j_2 - j_1|, \ldots, j_1 + j_2$, where j increases in steps of 1. *Given this fact, what are the possible values of j in the case of two spin-halfs? For each such j that you obtain, what are the possible values of m? Finally for each such value of m what are the possible values of m_1 and m_2?*

Hint: What did we find in 1e)?

(2 points)

b) *By the results in a) argue that there exist some complex numbers α, β with $|\alpha|^2 + |\beta|^2 = 1$, and γ, δ with $|\gamma|^2 + |\delta|^2 = 1$ such that*

$$\begin{aligned}\Big|j = 0, m = 0\Big\rangle =& \alpha\Big|m_1 = \frac{1}{2}, m_2 = -\frac{1}{2}\Big\rangle + \beta\Big|m_1 = -\frac{1}{2}, m_2 = \frac{1}{2}\Big\rangle \\ \Big|j = 1, m = 0\Big\rangle =& \gamma\Big|m_1 = \frac{1}{2}, m_2 = -\frac{1}{2}\Big\rangle + \delta\Big|m_1 = -\frac{1}{2}, m_2 = \frac{1}{2}\Big\rangle\end{aligned}$$

Moreover, express $\Big|j = 1, m = -1\Big\rangle$ and $\Big|j = 1, m = 1\Big\rangle$ in terms of the $|m_1, m_2\rangle$ basis.

(2 points)

c) *Determine the expansion of $|j = 0, m = 0\rangle$ and $|j = 1, m = 0\rangle$ in terms of the $|m_1, m_2\rangle$ up to a global phase factor.* In other words, you should determine α, β and γ, δ up to global phase factors.

Hint:

Note that $\boldsymbol{J}^2|j = 0, m = 0\rangle = 0$ and $\boldsymbol{J}^2|j = 1, m = 0\rangle = 2\hbar^2|j = 1, m = 0\rangle$. Moreover, (2) could be useful. Recall that we use a simplified notions, such that $J_{1-}J_{2+}\Big|m_1 = \frac{1}{2}, m_2 = -\frac{1}{2}\Big\rangle$ can be rewritten as $J_{1-}|j_1 = \frac{1}{2}, m_1 = \frac{1}{2}\rangle J_{2+}|j_1 = \frac{1}{2}, m_1 = -\frac{1}{2}\rangle$. Moreover, recall that if a ladder operator 'hits the roof', this results in zero, e.g. $J_{1+}|j_1 = \frac{1}{2}, m_1 = \frac{1}{2}\rangle = 0$. Analogously, it results in zero if it 'hits the floor', e.g. $J_{2-}|j_2 = \frac{1}{2}, m_2 = -\frac{1}{2}\rangle = 0$. I may also mention that the end result of these derivations is very nice and simple.

(4 points)

Remark: Since we have determined the expansion of $|j = 0, m = 0\rangle$, $|j = 1, m = 1\rangle$, $|j = 1, m = 0\rangle$ and $|j = 1, m = -1\rangle$ in terms of the $|m_1, m_2\rangle$ basis, it means that we have determined the Clebsch-Gordan coefficients for the addition of two spin-half systems. Also, one may note that there only is a single state with $j = 0$, namely $|j = 0, m = 0\rangle$, which os commonly referred to as the *singlet-state*. In comparison, there are three states $|j = 1, m = 1\rangle$, $|j = 1, m = 0\rangle$ and $j = 1, m = -1\rangle$, which are referred to as *triplet states*.