Quantum Mechanics

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WS 24/25

Sheet 14 **Saturday January 25 at 24:00**

1 Perturbation of a rotational degree of freedom

Consider a system with only a rotational degree of freedom. (Like a rigid rotor, or particle restricted to move on the surface of a sphere.) We assume that the Hamiltonian can be written[1]

$$H_0 = A\boldsymbol{L}^2 + B\hbar L_3, \quad A > B > 0, \tag{1}$$

where we have the *orbital* (so no half-integer angular quantum numbers) angular momentum operators $\boldsymbol{L}^2$ and L_3, and where A and B are some constants.

a) *Determine the ground state energy and the three first excited energies of (1), as well as the corresponding eigenstates.*

Hint: Recall what are the eigenvalues and joint eigenvectors of $\boldsymbol{L}^2$ and L_3, for orbital angular momenta.

(2 points)

b) Suppose that we add a perturbation $W = L_2^2$ (note the square), such that the perturbed Hamiltonian reads

$$H = H_0 + \lambda W = A\boldsymbol{L}^2 + B\hbar L_3 + \lambda L_2^2$$

For the eigenvalues in ***a)****, determine their perturbations to the first order in λ.* In other words, determine the expansion $E(\lambda) = E + \lambda E^{(1)} + O(\lambda^2)$ for these four eigenvalues, where E denotes the unperturbed eigenvalues of H_0.

Hint: Recall how one can express J_2 in terms of J_+ and J_-. How does J_+ and J_- act on the eigenstates from **a)**?

(4 points)

c) *Determine the first order perturbation of the ground state and first excited state.* Hence, determine the expansion $|\Psi(\lambda)\rangle = |\Psi\rangle + \lambda|\Psi^{(1)}\rangle + O(\lambda^2)$ for the ground state and the first excited state.

(4 points)

d) *Determine the second order correction to the first excited energy.* In other words, determine the expansion $E(\lambda) = E + \lambda E^{(1)} + \lambda^2 E^{(2)} + O(\lambda^3)$ for the first excited state.

(2 points)

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[1] Recall that L_1, L_2, L_3 often are denoted L_x, L_y, L_z.

2 Coupling of harmonic oscillators via a perturbation

Suppose that we have two non-interacting harmonic oscillators[2]

$$H_0 = \hbar\omega a_1^\dagger a_1 + \hbar\omega a_2^\dagger a_2, \quad \omega > 0,$$

where $a_1, a_1^\dagger$ are the annihilation and creation operators of the first oscillator, and $a_2, a_2^\dagger$ the annihilation and creation operators of the second oscillator.

a) *Determine the eigenvalues and the corresponding eigenspaces of H_0. What are the degeneracies (i.e. the dimensions of the eigenspaces)?* You can specify the eigenspaces by listing basis elements that span the eigenspaces.

Hint: What are the eigenbases and eigenvalues of each separate oscillator?

(3 points)

b) Assume a perturbation $W = a_1^\dagger a_2 + a_2^\dagger a_1$, i.e., the new Hamiltonian becomes

$$H = H_0 + \lambda W = \hbar\omega a_1^\dagger a_1 + \hbar\omega a_2^\dagger a_2 + \lambda a_1^\dagger a_2 + \lambda a_2^\dagger a_1.$$

To the first order in λ, what happens to the ground state energy and the ground state eigenspace of the unperturbed Hamiltonian?

(2 points)

c) *To the first order in λ, determine the corrections to the original first excited energy of the unperturbed Hamiltonian.* In other words, determine the new eigenvalues corresponding to the original first excited energy.

(3 points)

[2] Strictly speaking, the Hamiltonian of an oscillator is $\hbar\omega(a^\dagger a + \frac{1}{2}\hat{1})$, but we skip the term $\frac{1}{2}\hat{1}$ since it serves no purpose for our current investigation.