Quantum Mechanics

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WS 24/25

Sheet 2 Saturday October 19 at 24:00 Uhr

1 Linear operators do not generally commute

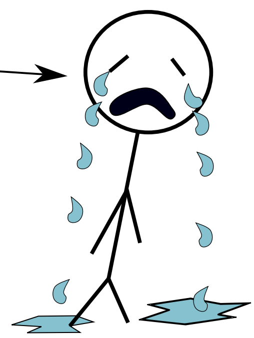
Consider the two matrices

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Calculate ZX and XZ. Is ZX equal to XZ?

(1 point)

Moral of this exercise: Do not swap the order of matrices (or linear operators) unless you can justify it!



2 Properties of observables

The main task of this sheet is to show the uncertainty relation, which we do in exercise 3. However, before we turn to that, we need to make a few preparations.

a) If A and B are two linear operators, the commutator is defined as $[A, B] = AB - BA$. Let $\alpha_k \in \mathbb{C}$ and A_k be operators. Similarly, let $\beta_k \in \mathbb{C}$ and let B_k be operators. *Show that*

$$[\sum_k \alpha_k A_k, B] = \sum_k \alpha_k [A_k, B], \quad [A, \sum_k \beta_k B_k] = \sum_k \beta_k [A, B_k].$$

This means that the commutator is linear in both its arguments (which is useful to remember).

(2 points)

b) We let X denote the position operator, which can be defined by it action on wave functions $\psi(x)$ as $(X\psi)(x) = x\psi(x)$. We also let P be the momentum operator, which we can define as $(P\psi)(x) = -i\hbar \frac{\partial}{\partial x}\psi(x)$. *Show the canonical commutation relation*

$$[X, P] = i\hbar\hat{1}.$$

Hint: Apply $[X, P]$ to a general wave function ψ. Then compare the result with $i\hbar\hat{1}\psi$. Here, $\hat{1}$ is the identity operator, which maps every function to itself, i.e., $(\hat{1}\psi)(x) = \psi(x)$. **(2 points)**

c) On spaces of (nice) wave functions we have the inner product $\langle \phi|\psi\rangle = \int_{-\infty}^{\infty} \phi(x)^* \psi(x) dx$. We say that an operator A is Hermitian if it satisfies $\langle \phi|A\psi\rangle = \langle A\phi|\psi\rangle$. In terms of wave functions, this translates to the requirement that $\int_{-\infty}^{\infty} \phi(x)^*(A\psi)(x)dx = \int_{-\infty}^{\infty} \left((A\phi)(x)\right)^* \psi(x)dx$ for all wave functions ϕ and ψ. *Show that the position operator X is Hermitian.* **(1 point)**

d) *Next, show that the momentum operator P is Hermitian. In other words, show that*

$$\int_{-\infty}^{\infty} \phi(x)^* \left(P\psi(x)\right) dx = \int_{-\infty}^{\infty} \left(P\phi(x)\right)^* \psi(x) dx.$$

Assume that $\lim_{x\to\pm\infty} \phi(x) = 0$ and $\lim_{x\to\pm\infty} \psi(x) = 0$.

Hint: Partial integration! **(2 points)**

e) Recall that a matrix A is Hermitian if we get back the same matrix again when we transpose it, and complex conjugate all its entries, i.e., if $A^\dagger = A$. At first sight, this may look very different from the definition of Hermiticity that we use in c) and d). However, consider the space of N-dimensional complex column vectors (column vectors with N entries). For two such column vectors b, c we can define the inner product as $\langle c|b\rangle = c^\dagger b = \sum_{n=1}^{N} c_n^* b_n$. Let A be an $N \times N$ matrix. *Show that the condition that* $\langle Ac|b\rangle = \langle c|Ab\rangle$ *for all* b, c *is equivalent to* $A^\dagger = A$. **(2 points)**

3 Heisenberg's uncertainty relation

Recall from the lecture that the variance of an observable A is defined as $\mathrm{Var}[A] = \langle (A - \langle A\rangle)^2\rangle$, where $\langle A\rangle = \int \psi(x)^* A\psi(x) dx$ denotes the expectation value. In this exercise we derive Heisenberg's uncertainty relation

$$\mathrm{Var}[X]\mathrm{Var}[P] \geq \frac{\hbar^2}{4}. \tag{1}$$

This inequality says that there can be no states that leads to probability distributions that are sharp in both position and momentum.[1]

a) Define the "centered" operators $\tilde{X} = X - \langle X\rangle \hat{1}$ and $\tilde{P} = P - \langle P\rangle \hat{1}$. *Show that*

$$\langle \tilde{X}\rangle = 0, \quad \langle \tilde{P}\rangle = 0, \quad \mathrm{Var}[\tilde{X}] = \mathrm{Var}[X], \quad \mathrm{Var}[\tilde{P}] = \mathrm{Var}[P], \quad [\tilde{X}, \tilde{P}] = [X, P].$$

(2 points)

Remark: One can think of the change to $\tilde{X}$ and $\tilde{P}$, as a change to a center of mass coordinate, where the origin is at the expectation value.

b) We now define the operator $\tilde{A} = \tilde{X} + i\lambda\tilde{P}$, where $\lambda \in \mathbb{R}$ is a free parameter that we can choose as we like. *Show that*

$$\int_{-\infty}^{\infty} |\tilde{A}\psi(x)|^2 dx = \mathrm{Var}[X] - \lambda\hbar + \lambda^2 \mathrm{Var}[P]. \tag{2}$$

Hint: Try to do as much as possible of the derivations by using the abstract operators and their properties and use the previous results. (In other words, *avoid* as far as possible to replace $X\psi$ with $x\psi(x)$, or $P\psi$ with $-i\hbar\frac{\partial}{\partial x}\psi(x)$). **(4 points)**

Remark: The operator $\tilde{A}$ is only a mathematical construction to aid the derivations, and has no direct physical meaning.

c) Denote the right hand side of (2) by $I(\lambda)$. *Explain why* $I(\lambda) \geq 0$. *Use this inequality, together with* (2), *in order to determine a lower bound on* $\mathrm{Var}[X]$. Your bound will hold for all choices of λ. *By using this freedom, find the strongest possible bound.* (This means that you should make the lower bound on $\mathrm{Var}[X]$ as large as you can.) *Use your result to derive* (1). **(4 points)**

[1] This is analogous to the time-frequency uncertainty relation for classical waves.