# Quantum Mechanics

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WS 24/25

**Sheet 5** **Saturday November 9 at 24:00**

## 1 Heisenberg's equations of motion and the evolution of expectation values

Suppose that we have three operators $Q_1, Q_2, Q_3$. Apart from these being Hermitian, we only know that they satisfy the following relations

$$[Q_1, Q_2] = iQ_3, \quad [Q_2, Q_3] = iQ_1, \quad [Q_3, Q_1] = iQ_2. \tag{1}$$

We moreover let the Hamiltonian of the system be

$$H = \alpha Q_3,$$

where $\alpha > 0$. Even though we don't know anything more about $Q_1, Q_2, Q_3$ than that they satisfy the relations (1), it turns out that we can determine how the expectation values $\langle Q_1\rangle(t), \langle Q_2\rangle(t), \langle Q_3\rangle(t)$ evolve.

**a)** *Show that the expectations values satisfies the equations of motion*

$$\frac{d}{dt}\begin{bmatrix}\langle Q_1\rangle(t)\\ \langle Q_2\rangle(t)\\ \langle Q_3\rangle(t)\end{bmatrix} = \boldsymbol{A}\begin{bmatrix}\langle Q_1\rangle(t)\\ \langle Q_2\rangle(t)\\ \langle Q_3\rangle(t)\end{bmatrix} \tag{2}$$

*and determine the* $3\times 3$ *matrix* $\boldsymbol{A}$.

**Hint:** Think of Heisenberg's equations of motions. Although we did not discuss it in the lecture, it can also be useful to have a look at the derivations of Ehrenfest's theorem in the lecture notes.

**(3 points)**

**b)** *Find the solution to* (2), *given the initial values* $\langle Q_1\rangle(0), \langle Q_2\rangle(0), \langle Q_3\rangle(0)$. *Can you describe the motion?*

**(2 points)**

**Remark:** The aspect that makes this derivation possible is that the Hamiltonian and the relevant observables are all part of a small closed commutator algebra. The latter means that if you commute two operators from such a collection of operators, you get back a linear combination of the operators in the same set. In the current case, (1) says that every commutation yields a particularly simple linear combination, where only one element in the collection is multiplied with a complex number.

## 2 Resonances in one-dimensional scattering

In the lecture we investigated tunnelling of a particle through a one-dimensional potential barrier. In this exercise, we investigate another phenomenon that occurs when the energy is high enough, such that we have plane-wave solutions everywhere. We consider the same potential as in the lecture

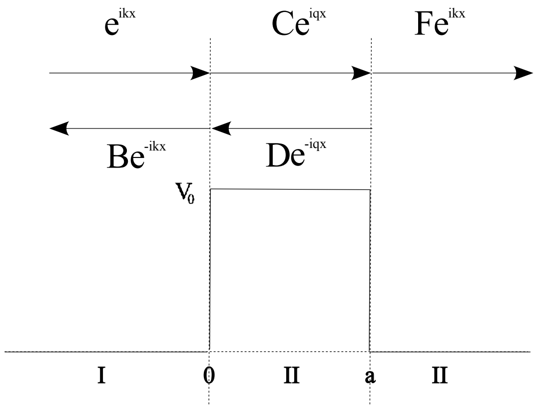
$$V(x) = \begin{cases} 0 & x < 0, \\ V_0 & 0 \le x \le a, \\ 0 & a < x, \end{cases}$$

but we assume energies

$$E > V_0,$$

which means that we have plane-wave solutions in all intervals. Recall that $e^{ikx}$ for $k > 0$ correspond to plane-waves moving from left to the right, while $e^{-ikx}$ moves from the right to the left. We here consider a setting where we send in a wave $e^{ikx}$ from the left (like in the lecture) in region I, and there is no wave moving in from the right ($e^{-ikx}$) in region III,

$$\begin{aligned} \phi_I(x) &= e^{ikx} + Be^{-ikx}, & x \le 0, \\ \phi_{II}(x) &= Ce^{iqx} + De^{-iqx}, & 0 \le x \le a, \\ \phi_{III}(x) &= Fe^{ikx}, & a \le x. \end{aligned}$$




**a)** *Express k and q in terms of $V_0$, the mass m and energy E (and ħ).* You do not have to derive this, it is enough to just state it.

**(1 point)**

**b)** *Use the relevant conditions for joining the wave functions, in order to show that the coefficients B, C, D and F are related by the equations*

$$\begin{aligned} C =& F\frac{1}{2}(1 + \frac{k}{q})e^{i(k-q)a} = \frac{1}{2}(1 + \frac{k}{q}) + B\frac{1}{2}(1 - \frac{k}{q}), \\ D =& F\frac{1}{2}(1 - \frac{k}{q})e^{i(k+q)a} = \frac{1}{2}(1 - \frac{k}{q}) + B\frac{1}{2}(1 + \frac{k}{q}). \end{aligned} \tag{3}$$

Note that there are four equations.

**Hint:** The joining results in four equations. From these, one can 'extract' $C$ and $D$ by adding and subtracting suitable pairs of equations.

**(4 points)**

**c)** Note that $F$ corresponds to the amplitude of the transmitted wave and $B$ to the reflected wave. In the following we wish determine these two amplitudes. *Show that*

$$B = \frac{(k^2 - q^2)\sin(qa)}{(q^2 + k^2)\sin(qa) + 2iqk\cos(qa)},$$

$$F = \frac{2iqke^{-ika}}{(q^2 + k^2)\sin(qa) + 2iqk\cos(qa)}.$$

**Hint:** Take a look at the equations in (3). One suitable way to eliminate $F$ (in order to get $B$) is to divide two of these equations with each other. After some rearrangements, one can do something similar for $B$.

**(5 points)**

**d)** The scattered particle is either reflected or transmitted, and correspondingly $|B|^2 + |F|^2 = 1$. *Verify that this is true in our case.* Consequently, $|F|$ and $|B|$ can at most be 1. There are some particular values for $q$ where the barrier becomes completely transparent. In other words, the particle just passes through, without any reflection, i.e., $|F| = 1$ and $|B| = 0$. *Determine the values of $q$, and correspondingly the values of $E$, for which $|T| = 1$ (and $B = 0$), where we assume that $E > V_0 > 0$.*

**(3 points)**

**e)** Now consider an infinite potential well, but where the 'bottom' of the well is at $V_0$. In other words

$$\tilde{V}(x) = \begin{cases} +\infty & x < 0, \\ V_0 & 0 \leq x \leq a, \\ +\infty & a < x. \end{cases}$$

*Determine the values of $q$ and $E$ where this system have an eigenstate. How does this compare with your results in d)?*

**(2 points)**

**Remark:** Whenever the particle hits a change of the potential, the wave-function is partially reflected. Because of the two sudden changes of the potential at the two interfaces, there can form standing waves in region II, at suitable energies. Such standing waves are the result of interferences between the various reflected waves. As it so happens, these interfere constructively in the forward direction, while negatively in the reflected direction, yielding the "perfect" transmission. This phenomenon is referred to as a "resonance".