Quantum Mechanics

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WS 24/25

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1 Harmonic oscillators and coherent states

As we have seen in the lecture, the Hamilton operator of the harmonic oscillator can be rewritten in terms of the annihilation and creation operators a and $a^\dagger$, a.k.a the ladder operators, as

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}\hat{1}), \tag{1}$$

where

$$a = \frac{1}{\sqrt{2}}(\tilde{X} + i\tilde{P}), \quad a^\dagger = \frac{1}{\sqrt{2}}(\tilde{X} - i\tilde{P}), \quad \tilde{X} = \sqrt{\frac{m\omega}{\hbar}}X, \quad \tilde{P} = \sqrt{\frac{1}{m\hbar\omega}}P.$$

We also introduced the normalized eigenstates ϕ_n of H,

$$H\phi_n = \hbar\omega(n + \frac{1}{2})\phi_n, \quad n = 0, 1, 2, \ldots,$$

where we equivalently could regard ϕ_n as the eigenstates, $N\phi_n = n\phi_n$, of the number operator $N = a^\dagger a$. By each application of the ladder operators we can, so to speak, step down or up a single rung of the 'ladder' of eigenstates

$$a\phi_0 = 0, \quad a\phi_n = \sqrt{n}\phi_{n-1}, \quad n = 1, 2, \ldots \quad \text{and} \quad a^\dagger\phi_n = \sqrt{n+1}\phi_{n+1}, \quad n = 0, 1, 2, \ldots.$$

a) The coherent states ψ_α are defined as the normalized states of the annihilation operator,

$$a\psi_\alpha = \alpha\psi_\alpha, \tag{2}$$

where it turns out to exist one such coherent state for each complex number $\alpha \in \mathbb{C}$. *Relate the real and imaginary parts of α to the expectation value $\langle\tilde{X}\rangle_\alpha = \langle\psi_\alpha|\tilde{X}\psi_\alpha\rangle$ of the dimensionless position operator $\tilde{X}$ and to the expectation value $\langle\tilde{P}\rangle_\alpha = \langle\psi_\alpha|\tilde{P}\psi_\alpha\rangle$ of the dimensionless momentum operator $\tilde{P}$.*

Hint: Recall that $\tilde{X}$ and $\tilde{P}$ are Hermitian operators, so their expectation values are real.

(2 points)

b) In the following, we wish to determine the variance of the position and momentum for the coherent states, and as a step towards this lofty goal: *Show that*

$$\langle\psi_\alpha|a^2\psi_\alpha\rangle = \alpha^2, \quad \langle\psi_\alpha|(a^\dagger)^2\psi_\alpha\rangle = (\alpha^*)^2, \quad \langle\psi_\alpha|a^\dagger a\psi_\alpha\rangle = |\alpha|^2, \quad \langle\psi_\alpha|aa^\dagger\psi_\alpha\rangle = |\alpha|^2 + 1.$$

Hint: Recall that generally $\langle\eta|C\eta\rangle^* = \langle C\eta|\eta\rangle$ and $\langle C\eta|\eta\rangle = \langle\eta|C^\dagger\eta\rangle$, which we can combine to $\langle\eta|C\eta\rangle^* = \langle\eta|C^\dagger\eta\rangle$. Recall also that the inner product $\langle\eta|\chi\rangle$ is *linear* in its second argument, such that $\langle\eta|z\chi\rangle = z\langle\eta|\chi\rangle$ for any complex number z, while it is *anti-linear* in its first argument so that $\langle z\eta|\chi\rangle = z^*\langle\eta|\chi\rangle$.

(4 points)

c) *Show that the coherent states satisfy the uncertainty relation* $\mathrm{Var}[X]\mathrm{Var}[P] \geq \hbar^2/4$ *with equality.*

Hint: Note that we here ask for the variances of X and P, rather than of $\tilde{X}$ and $\tilde{P}$, so keep in mind the relation between these.

Remark: This result means that the coherent states are minimum uncertainty wave-packets. In other words, for each point in phase space (the combination of position and momentum) the corresponding coherent state is as sharply focused as it can be. **(3 points)**

d) The definition (2) only implicitly defines the coherent states (but as we have seen above, one can get quite far with that). However, here we want to find an explicit description. More precisely, we wish to expand the coherent states ψ_α in terms of the orthonormal eigenbasis $\{\phi_n\}_{n=0,1,2,\ldots}$. *Determine the expansion coefficients* $f_n(\alpha)$ *and normalization constant* $C(\alpha)$*, such that*

$$\psi_\alpha = C(\alpha)\sum_{n=0}^{\infty} f_n(\alpha)\phi_n. \tag{3}$$

Hint: Recall that if $\sum_n c_n\phi_n = 0$ for a basis $\{\phi_n\}_n$, then $c_n = 0$. Try to find a recursion relation for $f_n(\alpha)$ and keep in mind that $a\phi_0 = 0$. **(4 points)**

e) *Determine the overlap* $|\langle\psi_\beta|\psi_\alpha\rangle|^2$ *for* $\alpha, \beta \in \mathbb{C}$. *Make sure that you find a closed expression, i.e., no infinite sums. Are two coherent states ever orthogonal to each other?*

(2 points)

f) In d) we have obtained an explicit description of the coherent states. However, it is also useful to know the wave-function. Equation (3) does actually give the wave-function, but only as an infinite sum of rather complicated objects. It turns out that the coherent states have rather simple wave-functions. In principle, we could evaluate the sum in (3), but that would require detailed knowledge about Hermite polynomials. We will instead follow an easier path, based on the expression $a = \frac{1}{\sqrt{2}}(\tilde{X} + i\tilde{P})$. Written in terms of differential operators, the defining equation $a\psi_\alpha = \alpha\psi_\alpha$ becomes

$$\frac{1}{\sqrt{2}}(\tilde{x} + \frac{d}{d\tilde{x}})\phi_\alpha = \alpha\phi_\alpha, \tag{4}$$

where we have used the identification[1] of $\tilde{P}$ with $-i\frac{d}{d\tilde{x}}$. *Show that the normalized solution of* (4) *takes the form*

$$\phi_\alpha(\tilde{x}) = \frac{1}{\pi^{1/4}}e^{-\frac{1}{2}\left(\tilde{x} - q\mathrm{Re}(\alpha)\right)^2 + ir\mathrm{Im}(\alpha)\tilde{x}}$$

and determine the real numbers q *and* r.

Hint: For the normalization, recall the Gaussian integral $\int_{-\infty}^{\infty} e^{-\alpha(\tilde{x}-\beta)^2}d\tilde{x} = \sqrt{\frac{\pi}{\alpha}}$. **(3 points)**

g) So far we have determined properties of the coherent states as such. Next, we wish to know how coherent states evolve in the Harmonic oscillator (1). *Show that if the Harmonic oscillator initially is put into a coherent state, then it remains in a coherent state, up to a global phase factor.*[2] *Moreover, for an initial coherent state* $\psi_{\alpha(0)}$*, determine the evolution* $\psi(t) = e^{i\chi(t)}\psi_{\alpha(t)}$ *in terms of the complex number* $\alpha(t)$ *and the (real) phase* $\chi(t)$.

Hint: Recall from the lecture how one can determine the time evolution of a state expanded in the eigenbasis of the Hamiltonian. **(2 points)**

[1] One may note that we identify P with $-i\hbar\frac{d}{dx}$, while the dimensionless operator $\tilde{P}$ is identified with $-i\frac{d}{d\tilde{x}}$, where $\tilde{x} = \sqrt{\frac{m\omega}{\hbar}}x$.

[2] Recall that two wave-functions that differ only by a global phase factor represent the same physical state.

Remark: Because of the uncertainty relation, there is no really direct quantum mechanical counterpart to the classical phase space. However, the coherent states do in some sense correspond to as sharp points in phase space as quantum mechanics allows. For each point in phase space there exists a coherent state with corresponding expectation values of position and momentum, and where the joint uncertainty of these observables is minimal.