Quantum Mechanics

David Gross, Johan Åberg

Institut für Theoretische Physik, Universität zu Köln

WS 24/25

Sheet 7 Saturday November 23 at 24:00

In this sheet we will use the Dirac notation (or bra-ket notation) quite extensively, where each element of the Hilbert space is denoted by a ket $|\psi\rangle$, and each dual vector is denoted by a bra $\langle\psi|$. Once getting used to it, it is a convenient notation.

1 The virtues of the completeness relation

For any orthonormal basis $\{|e_n\rangle\}_n$ the completeness relation (or resolution of identity) is

$$\sum_n |e_n\rangle\langle e_n| = \hat{1}.$$

This relation may look rather innocent, but it is actually quite useful. In particular, it allows us to transform between abstract objects in Hilbert spaces, and their representations in a basis. For example, recall from the lecture that for an operator Q, we have

$$Q = \hat{1}Q\hat{1} = \sum_n |e_n\rangle\langle e_n|Q\sum_{n'}|e_{n'}\rangle\langle e_{n'}| = \sum_{n,n'}|e_n\rangle Q_{n,n'}\langle e_{n'}|, \quad Q_{n,n'} = \langle e_n|Q|e_{n'}\rangle,$$

which thus directly tell how we should translate back and forth between operators and their matrix representations.

a) Suppose that we have a matrix representation $\boldsymbol{Q}^e$ of an operator Q with respect to the ON-basis $\{|e_n\rangle\}_n$. *How could we calculate the matrix representation $\boldsymbol{Q}^a$ with respect to another ON-basis $\{|a_k\rangle\}_k$?*

(2 points)

b) Let $\{|a_k\rangle\}_k$ and $\{|b_l\rangle\}_l$ be two orthonormal bases of the same Hilbert space. *Show that the matrix $\boldsymbol{U} = [\langle a_k|b_l\rangle]_{k,l}$ is unitary.*

(2 points)

c) Let Q be Hermitian operator. Let $\{|k\rangle\}_{k=1}^K$ be an orthonormal basis. We let the matrix $\boldsymbol{M}$ be the matrix representation of Q in the $\{|k\rangle\}_k$ basis. *Show that if the column vector $\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_K \end{bmatrix}$ is an eigenvector of $\boldsymbol{M}$ with eigenvalue λ, then $|\psi\rangle = \sum_{k=1}^K |k\rangle u_k$ is an eigenvector to Q with eigenvalue λ. Conversely, show that if $|\psi\rangle$ is an eigenvector to Q with eigenvalue λ, then the representation of $|\psi\rangle$ in the basis $\{|k\rangle\}_{k=1}^K$ is an eigenvector of $\boldsymbol{M}$ with eigenvalue λ.*

(2 points)

2 Unitary operators

a) Consider the following three operators

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|, \quad \sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|, \quad \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|.$$

where $\{|0\rangle, |1\rangle\}$ is an orthonormal basis. *Show that these three operators are both Hermitian and unitary.*

Hint: You can use whichever of the equivalent characterisations of unitary operators stated in the lecture notes. One of the standard characterisations becomes less painful to check if one uses the Hermiticity.

(3 points)

b) In the lecture notes it was claimed the all eigenvalues of a unitary operator has eigenvalue with absolute value 1. *Prove this claim. What is the consequence for the eigenvalues of an operator that is both Hermitian and unitary?*

(2 points)

c) Consider the operator σ_x from a). From a) we know that this operator is Hermitian. *Determine the eigenvalues, as well as the corresponding normalized eigenvectors, of σ_x. Express the eigenvectors in terms of $\{|0\rangle, |1\rangle\}$. How do your results compare with b)?*

Hint: Exercise 1 **c)** might be useful.

(3 points)

d) Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible transformation with a Jacobian that is equal to 1 at every point in $\mathbb{R}^n$. On the Hilbert space $L^2(\mathbb{R}^2)$ we consider the mapping U_Φ defined by

$$[U_\Phi \psi](x) = \psi\big(\Phi^{-1}(x)\big), \quad x \in \mathbb{R}^n.$$

Show that U_Φ is unitary. You can use any of the equivalent characterisations of unitary operators listed in the lecture notes.

Hint: Recall that for a change of variables, the Jacobian gives the change of the volume element in the integral.

(2 points)

3 Operator identities

For a Hermitian operator A, let $A|k\rangle = a_k|k\rangle$, where $\{|k\rangle\}_{k=1}^{K}$ is an orthonormal basis. We assume that A is non-degenerate, so that all the eigenvalues a_k are different.

a) *Show that*

$$(A - a_K\hat{1})(A - a_{K-1}\hat{1}) \cdots (A - a_2\hat{1})(A - a_1\hat{1}) = 0,$$

where '0' denotes the zero operator.

Hint: What happens with the different terms in the above product when you apply an eigenvector to A? Recall that the zero operator is the operator that maps every vector to the zero vector. Is there maybe som nice ON-basis that you could expand a general vector in?

(2 points)

b) *Show that*

$$\frac{A - a_K \hat{1}}{a_l - a_K} \frac{A - a_{K-1}\hat{1}}{a_l - a_{K-1}} \cdots \frac{A - a_{l+1}\hat{1}}{a_l - a_{l+1}} \frac{A - a_{l-1}\hat{1}}{a_l - a_{l-1}} \cdots \frac{A - a_2\hat{1}}{a_l - a_2} \frac{A - a_1\hat{1}}{a_l - a_1} = |l\rangle\langle l|. \tag{1}$$

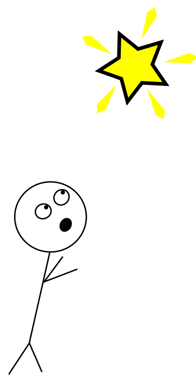
Note that the factor with $A - a_l$ is 'missing' in the above product. A more compact way to write the product in (1) is $\Pi_{k=1:k\neq l}^{K} \frac{A - a_k\hat{1}}{a_l - a_k}$.

Hint: As said previously, the completeness relation is a very convenient tool.

(2 points)

Remark: What (1) tells us is that the product on the left hand side is nothing but the projector onto the one-dimensional subspace spanned by the eigenvector $|l\rangle$. Now you might wonder why one should care about the monstrosity on the left hand side of (1), when one has the nice and simple expression on the right hand side. Equation (1) provides a method to explicitly express (the projectors onto) the eigenstates (in terms of the operator A and its eigenvalues), as opposed to implicitly determining them as the solutions to the equation $(A - a_l)|l\rangle = 0$. As a side remark, analogous relations still hold if there are degeneracies. We only assumed the non-degenerate case in order to simplify things.

4 Gold star exercise: Almost a resolution of identity for coherent states



This exercise gives absolutely no points, but if you want to know more about coherent states, this is the exercise for you!

In the previous exercise sheet you (hopefully) derived the expansion $\psi_\alpha = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \phi_n$ of the coherent state wave-function ψ_α in terms of the wave-functions ϕ_n of the eigenstates of the Harmonic oscillator Hamiltonian. In the following, we let $|n\rangle$ denote the eigenstates of the harmonic oscillator, i.e., $\phi_n(x) = \langle x|n\rangle$. We similarly let $|\alpha\rangle$ be the ket that corresponds to the coherent state, such that $\psi_\alpha(x) = \langle x|\alpha\rangle$. In terms of these kets, the expansion reads.

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

It turns out that the family of coherent states $\{|\alpha\rangle\}_{\alpha\in\mathbb{C}}$, although not forming an orthonormal basis, nevertheless satisfies kind of a resolution of identity

$$\int |\alpha\rangle\langle\alpha|\, d^2\alpha = \pi\hat{1}. \tag{2}$$

Hence, it is a resolution of identity, up to the extra factor π. Here, the integral $\int \cdots d^2\alpha$ means that we integrate the real and imaginary part of α as two independent real numbers.

The task is to prove (2).

Hint: Change to polar coordinates. In the radial part you can make another change of variables such that you reach the integral that defines the Gamma function $\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds$. One may recall that $\Gamma(m) = (m-1)!$

(0 points)

Remark: There is a generalization of the notion of bases that is referred to as 'frames', where this set of vectors can be used to represent general vectors (like we expand vectors in a basis) but without the requirement of linear independence. The relation (2) shows that the set of coherent states $\{|\alpha\rangle\}_{\alpha\in\mathbb{C}}$ is a 'tight' frame. The general condition for a set $\{|\psi_x\rangle\}_x$ to be a frame is that there exist constants $b \geq a > 0$ such that $b\hat{1} \geq \sum_x |\psi_x\rangle\langle\psi_x| \geq a\hat{1}$ (where the sum equally well could be an integral). In the case of tight frames, we have $b = a$, which makes the frame easier to work with. The concept of frames has been popular in signal analysis, but many of these ideas can also be applied in quantum mechanics.