Quantum Mechanics

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WS 24/25

Sheet 8 Saturday November 30 at 24:00

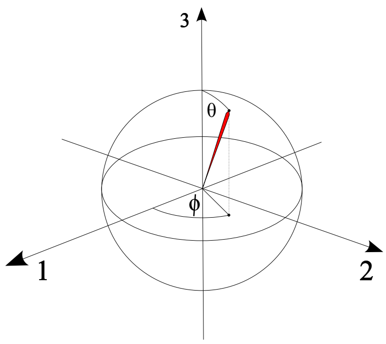
In the last sheet you got acquainted with the following three operators

$$\sigma_1 = \sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|, \quad \sigma_2 = \sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|, \quad \sigma_3 = \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|,$$

where $\{|0\rangle, |1\rangle\}$ is an orthonormal basis. Here, we introduce the notation $\sigma_1, \sigma_2, \sigma_3$ for $\sigma_x, \sigma_y, \sigma_z$, which is a common alternative, which also happens to be rather convenient for the things we are going to do in this sheet. These are named the Pauli operators, and are intimately related to two-dimensional Hilbert spaces, like that of spin-half particles.

1 The Bloch sphere

In general it is not easy to visualise quantum states. However, for two-dimensional Hilbert spaces there happens to exist an unusually nice representation, namely the Bloch sphere, and in this exercise we are going to introduce this tool.



a) For a two-dimensional Hilbert space, with basis $\{|0\rangle, |1\rangle\}$, we can reach all normalized vectors as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Show that, up to a global phase factor,[1] *we can write any such state as*

$$|\psi(\theta, \phi)\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi. \tag{1}$$

Hint: Try polar representations of the two complex numbers α and β.

(3 points)

b) We can interpret $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$ as the polar and azimuthal angles, respectively, in a spherical coordinate system (with radial coordinate $r = 1$). In other words, we can map the set of two-level states onto a spherical surface. Recall that the mapping to the Cartesian coordinates x_1, x_2, x_3 is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}. \tag{2}$$

[1] Recall that two vectors that only differ by a global phase factor represent the same state. Just to clarify the terminology, $e^{i\varphi}$ is a "phase factor´´ and φ is the "phase".

In turns out that these Cartesian coordinates is nothing but the expectation values of the Pauli operators. *More precisely, show that*

$$\langle\psi|\sigma_1|\psi\rangle = x_1, \quad \langle\psi|\sigma_2|\psi\rangle = x_2, \quad \langle\psi|\sigma_3|\psi\rangle = x_3,$$

for $|\psi\rangle$ *in* (1) *and* x_1, x_2, x_3 *as in* (2).

(3 points)

2 Eigenstates of a spin-half particle in a magnetic field

To a spin-half particle we can typically associate a magnetic moment, i.e., the particle acts as a small magnet, which can be affected by an external magnetic field. We consider a homogeneous magnetic field in the direction of a unit vector $\vec{\omega} \in \mathbb{R}^3$, i.e. $\sum_{j=1}^{3} \omega_j^2 = 1$. The Hamiltonian of the spin-half particle in this field we write as

$$H = -\frac{\gamma}{2}\vec{\omega} \cdot \vec{\sigma}, \qquad \gamma > 0, \tag{3}$$

Recall from problem 2a) on the last sheet that the Pauli operators are Hermitian. Since $\vec{\omega} \in \mathbb{R}^3$, it follows that the linear combination $\vec{\omega} \cdot \vec{\sigma} = \sum_{j=1}^{3} \omega_j \sigma_j$, and thus also H.

a) One may note that in the special case where the magnetic field is directed as $\vec{\omega} = (0, 0, 1)$, i.e., along the 3-axis (or z-axis) then

$$H = -\frac{\gamma}{2}\sigma_3 = -\frac{\gamma}{2}|0\rangle\langle 0| + \frac{\gamma}{2}|1\rangle\langle 1|.$$

What is the ground state (corresponding to the lowest energy) and what is the excited state (corresponding to the highest energy) of this Hamiltonian? What points do these two states correspond to on the Bloch sphere? How do these two points compare with the direction of the magnetic field $\vec{\omega} = (0, 0, 1)$*?*

(2 points)

b) Now suppose that the magnetic field is oriented in the direction

$$\vec{\omega}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Confirm that $|\psi(\theta, \phi)\rangle$ *and* $|\psi(\pi - \theta, \phi + \pi)\rangle$ *are eigenstates to* $H(\theta, \phi)$ *and determine the corresponding eigenvalues.* The eigenstates $|\psi(\theta, \phi)\rangle$ and $|\psi(\pi - \theta, \phi + \pi)\rangle$ correspond to two points on the Bloch sphere. *How does these two points relate to each other geometrically? Moreover, how do these two points relate to the direction* $\vec{\omega}(\theta, \phi)$ *of the magnetic field?* For your derivations, you can use that $|\psi(\pi - \theta, \phi + \pi)\rangle$ can be rewritten as

$$|\psi(\pi - \theta, \phi + \pi)\rangle = \sin(\theta/2)|0\rangle - e^{i\phi}\cos(\theta/2)|1\rangle,$$

as well as the trigonometric identities

$$\sin(\theta/2)\sin\theta + \cos(\theta/2)\cos\theta = \cos(\theta/2), \quad \cos(\theta/2)\sin\theta - \sin(\theta/2)\cos\theta = \sin(\theta/2).$$

Hint: Note that since you are given candidate eigenstates, you only need to *confirm* that these actually are solutions to the eigenvalue equation (i.e., just plug them in and test it). There is no need to employ the full machinery of finding solutions to the eigenvalue equations. Moreover, if you compare with problem a), I believe that you can guess which of $|\psi(\theta, \phi)\rangle$ and $|\psi(\pi - \theta, \phi + \pi)\rangle$ that correspond to the ground state.

(4 points)

3 Dynamics of a spin-half particle in a magnetic field

In the previous exercise we considered the eigenvalues and eigenstates of the spin-half particle. Here, we turn to dynamics of this system.

a) Before turning to the evolution operator, we need a technical observation. *Show that*

$$(\vec{\omega} \cdot \vec{\sigma})^2 = \hat{1},$$

where you are allowed to use the relation

$$\sigma_j \sigma_k = \delta_{jk} \hat{1} + i \sum_l \epsilon_{jkl} \sigma_l.$$

Here ϵ_{jkl} is the Levi-Civita symbol, which is anti-symmetric with respect to all permutations of the indices (e.g. $\epsilon_{jkl} = -\epsilon_{kjl}$).

(3 points)

b) Recall that the time evolution is given by the family of unitary operators $U(t) = e^{-itH/\hbar}$, for the Hamiltonian (3). *Show that*

$$U_{\vec{\omega}}(t) = \cos\left(\frac{t\gamma}{2\hbar}\right) \hat{1} + i \sin\left(\frac{t\gamma}{2\hbar}\right) \vec{\omega} \cdot \vec{\sigma}.$$

Hint: Consider Taylor expansions.

(3 points)

c) Let us now consider how the evolution looks on the Bloch sphere. For the sake of simplicity, we assume that the magnetic field is directed along the 3-axis (the z-axis), i.e.,

$$U_{(0,0,1)}(t) = \cos\left(\frac{t\gamma}{2\hbar}\right) \hat{1} + i \sin\left(\frac{t\gamma}{2\hbar}\right) \sigma_z.$$

For an initial state $|\psi(\theta, \phi)\rangle$, determine $U_{(0,0,1)}(t)|\psi(\theta, \phi)\rangle$. Show that the corresponding Bloch vector rotates around the 3-axis (z-axis), and determine the rotation rate.

Hint: Recall that a global phase shift does not affect the Bloch vector.

(2 points)

Remark: A spin (with magnetic moment) in a constant uniform magnetic field precesses, i.e., the component orthogonal to the field rotates. We can conclude that for this system, the Hamiltonian evolution induces rotations. We can additionally conclude that the unitary operators $U(\alpha) = e^{i\alpha\vec{\omega}\cdot\vec{\sigma}}$ represent rotations around the $\vec{\omega}$-axis. This provides an example of how families of unitary operators can represent symmetry operations. One may also note that the operator $\vec{\omega} \cdot \vec{\sigma}$ generates the family of these unitary representations, via the exponentiation.