Quantum Mechanics

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Sheet 9 Saturday December 7 at 24:00

1 Complex conjugation as time-reversal

In the lecture we discussed that there are not only unitary operators, but also such things as anti-unitary operators. An operator $Q : \mathcal{H} \to \mathcal{H}$ is called *anti-unitary* if it is *invertible* and satisfies

$$\langle Q\phi | Q\psi \rangle = \langle \phi | \psi \rangle^*, \quad \forall \phi, \psi \in \mathcal{H}.$$

a) We can regard the complex conjugation of the wave-function as an operator $C : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined by

$$(C\psi)(x) = \psi(x)^*.$$

What is the inverse C^{-1}? Show that C is anti-unitary. **(2 points)**

b) One has to be careful when dealing with anti-unitary operators. For invertible *linear* operators L, it is the case that these commute with all complex numbers, in the sense that $L(z\hat{1})L^{-1} = z\hat{1}$ for all $z \in \mathbb{C}$. However, for C it is the case that $C(i\hat{1})C^{-1} = -i\hat{1}$. *Show the latter by demonstrating that*

$$\big(C(i\hat{1})C^{-1}\psi\big)(x) = -i\psi(x), \tag{1}$$

for all wave-functions ψ. If we instead would have a real number $r \in \mathbb{R}$, what would be the result of $\big(C(r\hat{1})C^{-1}\psi\big)(x)$?

Hint: The left hand side of (1) may perhaps be a bit confusing at first sight. It might help to introduce the new wave-function $\eta(x) = i(C^{-1}\psi)(x)$, and realise that the left hand side of (1), is nothing but $(C\eta)(x)$. **(2 points)**

c) Next, we consider the action of C on operators. Let us consider a particle on a line $\mathbb{R}$, with position operator X and momentum operator P. *Show that*

$$(CXC^{-1}\psi)(x) = (X\psi)(x), \quad (CPC^{-1}\psi)(x) = (-P\psi)(x). \tag{2}$$

(2 points)

Remark: We can conclude that the position operator is unaffected, but the momentum operator switches sign under the action of C.

d) Now we turn to the question of why C can be regarded as reversing time-evolution. Suppose that a Hamiltonian H is *time-reversal symmetric*, meaning that

$$CHC^{-1} = H.$$

Show that the family of evolution operators $U(t) = e^{-itH/\hbar}$ satisfies

$$CU(t)\psi(x) = U(-t)C\psi(x). \tag{3}$$

for all ψ. Note that I in this expression have avoided to include as many brackets as above, since it would get somewhat unbearable to read. **(3 points)**

Remark: From (3) one can see that the application of C somehow swaps the direction of evolution from t to $-t$. This reversal may become a bit more manifest with some manipulations of (3). If we substitute t with $-t$ and then apply C to both sides, the result is

$$U(-t)\psi = CU(t)C\psi. \tag{4}$$

This formula says that $CU(t)C$ is the same as the backwards evolution $U(-t)$. Substituting ψ with $U(t)\psi$ in (4) yields a third option, namely $\psi = CU(t)CU(t)\psi$. This means that if you evolve forward in time by t, apply C, evolve by t, and finally apply C, you are back where you started. The conclusion is that in a time-reversal symmetric system, the application of C makes the system behave as if time run backwards. We can thus interpret the complex conjugation C as implementing time-reversal.

e) One should note that the time-reversal effect described above really requires that the Hamiltonian is time-reversal symmetric. However, not all Hamiltonians are. *Determine which of these two Hamiltonians are time-reversal symmetric or not*

$$H_1 = \frac{1}{2m}P^2 + V(X), \quad H_2 = \frac{1}{2m}P^2 + qXP + qPX.$$

Here V is a potential (with a nice Taylor expansion), m is a mass and q a real number.

Hint: Note that we can rewrite (2) as $CXC^{-1} = X$ and $CPC^{-1} = -P$. **(2 points)**

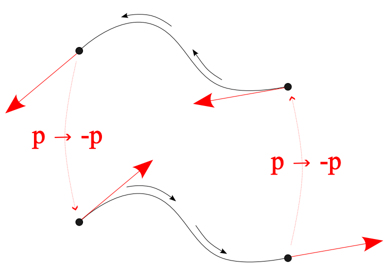
Remark: The canonical example of a system that breaks time-reversal symmetry is a charged particle in an external magnetic field.

f) The effect of the time-reversal can also be observed directly in the Schrödinger equation. Let $\psi(x,t)$ be a solution to the time-dependent Schrödinger equation

$$-i\hbar\frac{\partial\psi}{\partial t}(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial^2 x}(x,t) + V(x)\psi(x,t). \tag{5}$$

Show that $\phi(x,t) = \psi(x,-t)^*$ also is a solution to (5). **(3 points)**

Remark: Time-reversal and time-reversal symmetry is not an exclusively quantum mechanical phenomenon; there is a classical counterpart. As you may recall, the arena of classical mechanics is the phase space, which combines the space coordinates x and the momentum coordinates p. In this classical setting, we introduce a time-reversal operation with the effect of reversing all momenta, while keeping all positions intact $(x,p) \to (x,-p)$. Suppose that the Hamilton function is time-reversal symmetric, here meaning that $H(x,-p) = H(x,p)$. Then, if we evolve the particle for some time t, reverse the momentum of each particle ($p \to -p$, but leave their positions intact), evolve for some time t again, and finally switch the direction of all the momenta again ($p \to -p$), then we are back at the same position and the same momentum as where we started (thus at the same position in phase space). Contemplate on how all of this compare with the quantum setting, e.g. (2) and $\psi = CU(t)CU(t)\psi$.



2 Baker-Campbell-Hausdorff

As we have seen in the previous exercise, operator exponentiation is a central task. In the previous exercise, though, we were lucky in that the operator exponential could be obtained in a closed form. The situation is usually not as favourable as that. However, for certain tasks, it can be sufficient to only determine the exponentiation up to some degree of approximation. The Baker-Campbell-Hausdorff (BCH) formula is an important tool for such cases. To put this in perspective, one might recall that for numbers it is the case that $e^{a+b} = e^a e^b$.

a) For operators, we typically have $e^{A+B} \neq e^A e^B$. The general exception is if two operators commute. So, before turning to BCH, let us consider the commutative case. *Show that if* $[A, B] = 0$, *then* $e^{A+B} = e^A e^B$. Note that we are *not* assuming that A and B are diagonalizable, only that they commute.

Hint: Expansions are your friends. Since A and B commute, products can be rearranged in a nice form. In particular, think of $(A + B)^n$. What would the expansion of this be in case that A and B were real numbers. How does this compare with the case when A and B are commuting operators?

(3 points)

b) Now we turn to the case of two not necessarily commuting operators and the BCH formula.[1] *Show that*

$$e^{tA}e^{tB} = e^{tA+tB+\frac{t^2}{2}[A,B]} + O(t^3).$$

Hint: Expand both sides up to the second order in t and compare terms.

(3 points)

[1]The more general BCH expansion is an infinite sum of nested commutators, but here we terminate this expansion already at the second order.