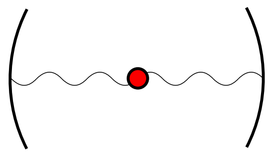
Dynamics of the Jaynes-Cummings model

Lecture November 26, 2019

1 The JC-model



In the previous lecture we derived the Jaynes-Cummings model, which describes the resonant interaction between two energy levels of an atom, and one single mode in a cavity. The model is described by the Hamilton operator

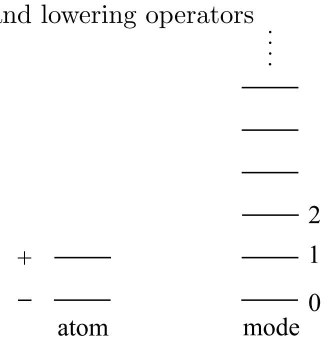
$$H = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega_0 a^\dagger a + \hbar g\sigma_+ a + \hbar g\sigma_- a^\dagger,$$

where a and $a^\dagger$ are the annihilation and creation operators of one bosonic mode. The Hamiltonian of the atomic two-level system is described by the σ_z-operator

$$\sigma_z = |+\rangle\langle+| \; - \; |-\rangle\langle-|.$$

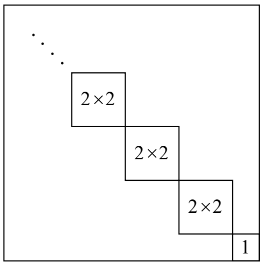
On the atom we furthermore have the the raising and lowering operators

$$\sigma_+ = |+\rangle\langle-|,$$
$$\sigma_- = |-\rangle\langle+|.$$



2 The evolution operator

In order to investigate the dynamics of the JC-model, we first determine the evolution operator $U(t) = e^{-itH/\hbar}$. We will do this via diagonalization of H. If we have an eigenvalue decomposition $H = \sum_k a_k |a_k\rangle\langle a_k|$, it follows that $U(t) = \sum_k e^{-ita_k/\hbar}|a_k\rangle\langle a_k|$. It is far from self-evident that we would be able to find an eigenvalue decomposition, especially in this case, since the underlying Hilbert space is infinite-dimensional. However, it turns out the the JC-Hamiltonian has a particularly simple structure, since it is block-diagonal with respect to the $|\pm, n\rangle$-basis. Our first step is to find this block-diagonalization, and next, to diagonalize these blocks.



Our first observation is that the ground state of the atom, and the vacuum state of the mode, is an eigenstate of the Hamiltonian

$$H|-,0\rangle = -\frac{\hbar\omega_0}{2}|-,0\rangle.$$

This yields a 1×1 block in the decomposition. Next, we note that

$$\begin{aligned} H|+,n-1\rangle =& \Big(\frac{\hbar\omega_0}{2} + \hbar\omega_0(n-1)\Big)|+,n-1\rangle + \hbar g\sqrt{n}|-,n\rangle, \\ H|-,n\rangle =& \Big(-\frac{\hbar\omega_0}{2} + \hbar\omega_0 n\Big)|-,n\rangle + \hbar g\sqrt{n}|+,n-1\rangle. \end{aligned} \tag{1}$$

This means that H maps all elements of the subspace spanned by $\{|+,n-1\rangle, |-,n\rangle\}$ back to that subspace. In other words, H block-diagonalizes with respect to these subspaces. From (1) we can determine the matrix elements of the corresponding block-matrices

$$\begin{bmatrix} \langle +,n-1|H|+,n-1\rangle & \langle +,n-1|H|-,n\rangle \\ \langle -,n|H|+,n-1\rangle & \langle -,n|H|-,n\rangle \end{bmatrix} = \begin{bmatrix} 0 & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & 0 \end{bmatrix} + \hbar\omega_0(n-\frac{1}{2})\hat{1}.$$

The eigenvalues and eigenvectors of this matrix are

$$E_{n,\pm} = \pm\hbar g\sqrt{n} + \hbar\omega_0(n-\frac{1}{2}), \quad |\psi_{n,\pm}\rangle := \frac{1}{\sqrt{2}}(|+,n-1\rangle \pm |-,n\rangle).$$

The evolution operator can thus be obtained as

$$\begin{aligned} U(t) =& e^{-itH/\hbar} \\ =& e^{\frac{1}{2}it\omega_0}|-,0\rangle\langle -,0| \\ &+ \sum_{n=1}^{\infty} e^{-it\omega_0(n-\frac{1}{2})}e^{-itg\sqrt{n}}|\psi_{n,+}\rangle\langle\psi_{n,+}| \\ &+ \sum_{n=1}^{\infty} e^{-it\omega_0(n-\frac{1}{2})}e^{itg\sqrt{n}}|\psi_{n,-}\rangle\langle\psi_{n,-}| \\ \\ =& e^{\frac{1}{2}it\omega_0}|-,0\rangle\langle -,0| \\ &+ \sum_{n=1}^{\infty} e^{-it\omega_0(n-\frac{1}{2})}\Big[\cos(tg\sqrt{n})\Big(|+,n-1\rangle\langle +,n-1| + |-,n\rangle\langle -,n|\Big) \\ &\qquad - i\sin(tg\sqrt{n})\Big(|+,n-1\rangle\langle -,n| + |-,n\rangle\langle +,n-1|\Big)\Big]. \end{aligned}$$

We have thus managed to determine the evolution operator of the JC-model.

3 Dynamics

Now that we have found the evolution operator, we can start to explore the dynamics of the JC-model. Let us see what happens if we start the atom in the ground state $|-\rangle$, and the mode in a number-state $|m\rangle$, for $m > 0$,

$$\begin{aligned} U(t)|-,m\rangle =& e^{\frac{1}{2}it\omega_0}|-,0\rangle\langle -,0|-,m\rangle \\ &+ \sum_{n=1}^{\infty} e^{-it\omega_0(n-\frac{1}{2})}\Big[\cos(tg\sqrt{n})\Big(|+,n-1\rangle\langle +,n-1|-,m\rangle \ + \ |-,n\rangle\langle -,n|-,m\rangle\Big) \\ &\qquad - i\sin(tg\sqrt{n})\Big(|+,n-1\rangle\langle -,n|-,m\rangle \ + \ |-,n\rangle\langle +,n-1|-,m\rangle\Big)\Big] \\ =& e^{-it\omega_0(m-\frac{1}{2})}\Big[\cos(tg\sqrt{m})|-,m\rangle - i\sin(tg\sqrt{m})|+,m-1\rangle\Big]. \end{aligned}$$

One may observe that this is an entangled state of the atom and the mode. If we wish to find the state of the atom alone, we need to determine the reduced density operator,

$$\begin{aligned} \rho_{\text{atom}} =& \text{Tr}_{\text{mode}}\Big(U(t)|-,m\rangle\langle -,m|U(t)^{\dagger}\Big) \\ =& \cos^2(tg\sqrt{m})\text{Tr}_{\text{mode}}(|-,m\rangle\langle -,m|) \\ &+ i\cos(tg\sqrt{m})\sin(tg\sqrt{m})\text{Tr}_{\text{mode}}(|-,m\rangle\langle +,m-1|) \\ &- i\cos(tg\sqrt{m})\sin(tg\sqrt{m})\text{Tr}_{\text{mode}}(|+,m-1\rangle\langle -,m|) \\ &+ \sin^2(tg\sqrt{m})\text{Tr}_{\text{mode}}(|+,m-1\rangle\langle +,m-1|) \\ =& \sin^2(tg\sqrt{m})|+\rangle\langle +| + \cos^2(tg\sqrt{m})|-\rangle\langle -|. \end{aligned}$$

If $P_+(t)$ denotes the probability that we find the atom in the excited state, then

$$P_+(t) = \sin^2(tg\sqrt{m}).$$

This describes an oscillatory motion of the energy, where one quantum moves between the atom and the mode. Even though only one quantum oscillates, the speed of the oscillations nevertheless depends on the number of excitations in the mode. This is the effect of stimulated emission and absorption. In principle, we can thus count the number of photons in the mode, if we could monitor the evolution of P_+.

4 Semi-classical model

For the sake of comparison, we shall here consider a simpler semi-classical model of a two-level system, where we do not explicitly model any degrees of freedom of the electromagnetic field, but only model the effect of it in terms of a time-dependent Hamiltonian

$$H = \frac{\hbar\omega_0}{2}\sigma_z + \gamma e^{-it\omega_0}\sigma_+ + \gamma e^{it\omega_0}\sigma_-.$$

This results in the family of evolution operators

$$\tilde{U}(t) = \cos(\Omega t)\Big[|+\rangle\langle +| \ + \ |-\rangle\langle -|\Big] - i\sin(\Omega t)\Big[|+\rangle\langle -| \ + \ |-\rangle\langle +|\Big], \qquad \Omega := \frac{\gamma}{\hbar}.$$

Hence, if $|\psi(0)\rangle = |-\rangle$, then

$$|\psi(t)\rangle = -i\sin(\Omega t)|+\rangle + \cos(\Omega t)|-\rangle. \tag{2}$$

Hence, the state oscillates between the ground state and the excited state. More importantly, in between, we have a *superposition* between the ground state and excited state.

5 Comparison

Let us now compare the dynamics that we obtained for the JC-model in section 3 with thee one we got for the semi-classical model. From (2) the semi-classical state can be rewritten in terms of the corresponding density operator as

$$\begin{aligned}\rho_{\text{atom}}^{\text{semi-classical}} =& |\psi(t)\rangle\langle\psi(t)| \\ =& \sin^2(\Omega t)|+\rangle\langle+| \;+\; \cos^2(\Omega t)|-\rangle\langle-| \\ &+ i\sin(\Omega t)\cos(\Omega t)\Big[|-\rangle\langle+| \;-\; |+\rangle\langle-|\Big].\end{aligned}$$

However, for the JC-model we have

$$\rho_{\text{atom}}^{\text{JC}} = \sin^2(tg\sqrt{m})|+\rangle\langle+| \;+\; \cos^2(tg\sqrt{m})|-\rangle\langle-|.$$

Hence, there is no off-diagonal element in the JC-model, which is quite a difference compared to the semi-classical model. This looks a bit worrying.

6 Semi-classical limit in the JC model

As was suggested in previous lectures, the semi-classical model can emerge when the fields are in coherent states. Unfortunately, it is not as simple as getting back the semi-classical model just by plugging in coherent states. Roughly speaking, one could expect things to work out when the amplitude of the coherent state is large, and the coupling between the atom and the mode is weak. Here we will attempt to make this idea more concrete, by considering the limit

$$g = \frac{\theta}{|\alpha|}, \quad |\alpha| \to \infty,$$

where θ is a constant. In what follows we will argue (in a not a particularly rigorous manner) that we can regain a unitary evolution on the atom in this limit. Let us first recall the form of coherent states

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle.$$

One particular property of coherent states is that the particle number is Poisson distributed. In other words, the probability to find n photons in the mode is

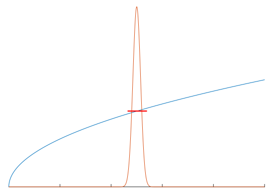
$$p_n = |\langle n|\alpha\rangle|^2 = e^{-\gamma}\frac{\gamma^n}{n!}, \qquad \gamma = |\alpha|^2.$$

Our goal is to determine how the atom evolves when it interacts with the coherent state. Hence, we wish to determine the reduced density operator of the atom

$$\begin{aligned}\rho_{\text{atom}}(t) =& \operatorname{Tr}_{\text{mode}}\Big(U(t)[\rho \otimes |\alpha\rangle\langle\alpha|]U(t)^\dagger\Big) \\ =& \sum_n e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} V_g(n)\rho V_g(n)^\dagger,\end{aligned}$$

where

$$\begin{aligned}V_g(n) =& e^{-it\omega_0 \frac{1}{2}} \cos(tg\sqrt{n+1})|+\rangle\langle+| \\ &+ e^{it\omega_0 \frac{1}{2}} \cos(tg\sqrt{n})|-\rangle\langle-| \\ &- ie^{-it\omega_0 \frac{1}{2}} \sin(tg\sqrt{n+1}) \frac{\alpha}{\sqrt{n+1}}|+\rangle\langle-| \\ &- ie^{it\omega_0 \frac{1}{2}} \sin(tg\sqrt{n}) \frac{\sqrt{n}}{\alpha}|-\rangle\langle+|.\end{aligned}$$



There are all these annoying square roots. If we could get rid of them (and the α), then we would have a unitary evolution. To this end, let us make a couple of observations. First, the square root $\sqrt{x}$ gets more and more flat the larger x is (the derivative goes to zero). Second, the Poisson distribution is approximately equal to the normal distribution $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$, for large values of $|\alpha|$. Moreover, the approximate normal distribution has mean $|\alpha|^2$ and standard deviation $|\alpha|$. This means that the width of the distribution grows much slower than the mean. Hence, for large $|\alpha|$ it seems reasonable that the square root is approximately constant on the region where the Poisson/normal distribution is large. To understand the following approximate reasoning, it might help to keep these observations in mind.

$$\begin{aligned}\rho_{\text{atom}}(t) =& \sum_n e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} V(t,n)\rho V(t,n)^\dagger \\ & \text{[Approximating the weighted sum over the Poisson distribution} \\ & \text{with the integral weighted with the normal distribution} \\ & f_{|\alpha|^2,|\alpha|}(x) \text{ with mean } \mu = |\alpha|^2 \text{ and standard deviation } \sigma = |\alpha|.] \\ \approx& \int f_{|\alpha|^2,|\alpha|}(x) V(t,x)\rho V(t,x)^\dagger dx \\ & [\quad \text{We cut away the tails of the integral at } s \text{ standard deviations.} \quad] \\ \approx& \int_{|\alpha|^2 - s|\alpha|}^{|\alpha|^2 + s|\alpha|} f_{|\alpha|^2,|\alpha|}(x) V(t,x)\rho V(t,x)^\dagger dx \\ & [\quad \text{Change of variables } x = y|\alpha| + |\alpha|^2 \quad] \\ =& \int_{-s}^{+s} f_{0,1}(y) V(y|\alpha| + |\alpha|^2)\rho V(y|\alpha| + |\alpha|^2)^\dagger dy.\end{aligned} \tag{3}$$

Next, we use our assumption

$$g = \frac{\theta}{|\alpha|}.$$

For the sake of simplicity we assume that α is real, i.e., $\alpha = |\alpha|$. Then

$$\begin{aligned}
V_{\theta/|\alpha|}(x|\alpha| + |\alpha|^2) =& e^{-it\omega_0 \frac{1}{2}} \cos\left(t\theta\sqrt{1 + \frac{x}{|\alpha|} + \frac{1}{|\alpha|^2}}\right) |+\rangle\langle+| \\
&+ e^{it\omega_0 \frac{1}{2}} \cos\left(t\theta\sqrt{1 + \frac{x}{|\alpha|}}\right) |-\rangle\langle-| \\
&- ie^{-it\omega_0 \frac{1}{2}} \sin\left(t\theta\sqrt{1 + \frac{x}{|\alpha|} + \frac{1}{|\alpha|^2}}\right) \frac{1}{\sqrt{1 + \frac{x}{|\alpha|} + 1}} |+\rangle\langle-| \\
&- ie^{it\omega_0 \frac{1}{2}} \sin\left(t\theta\sqrt{1 + \frac{x}{|\alpha|}}\right) \sqrt{1 + \frac{x}{|\alpha|}} |-\rangle\langle+|.
\end{aligned}$$

Keep in mind that $s \geq x \geq -s$, with s constant. Hence

$$\begin{aligned}
\tilde{U}(t) :=& \lim_{|\alpha|\to\infty} V_{\theta/|\alpha|}(x|\alpha| + |\alpha|^2) \\
=& e^{-it\omega_0 \frac{1}{2}} \cos(t\theta) |+\rangle\langle+| + e^{it\omega_0 \frac{1}{2}} \cos(t\theta) |-\rangle\langle-| \\
&- ie^{-it\omega_0 \frac{1}{2}} \sin(t\theta) |+\rangle\langle-| - ie^{it\omega_0 \frac{1}{2}} \sin(t\theta) |-\rangle\langle+|.
\end{aligned}$$

As one can see, $\tilde{U}(t)$ is unitary. If we put this limit back into our approximate expression for ρ_{atom} in (3), we thus get

$$\begin{aligned}
\rho_{\text{atom}}(t) \approx& \int_{-s}^{+s} f_{0,1}(y) V(y|\alpha| + |\alpha|^2) \rho V(y|\alpha| + |\alpha|^2)^\dagger dy \\
\approx& \tilde{U}(t)\rho\tilde{U}(t)^\dagger \int_{-s}^{+s} f_{0,1}(y) dy \\
\approx& \tilde{U}(t)\rho\tilde{U}(t)^\dagger.
\end{aligned}$$

Hence, this line of reasoning suggests that in the joint limit of large amplitude and weak interactions we obtain unitary evolution on the atom.

7 Rabi oscillations: Decay and revival

So far we have looked at a particular large $|\alpha|$-limit, but what happens for finite (but still large) values of $|\alpha|$? Suppose that we start the atom in the ground state $|-\rangle$, and the field in a coherent state $|\alpha\rangle$. The probability $P_+(t)$ that we find the atom in the excited state at time t is

$$\begin{aligned}
P_+(t) =& \sum_n |\langle +, n|U(t)|-, \alpha\rangle|^2 \\
=& \sum_{n=1}^{\infty} \sin^2(tg\sqrt{n}) e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \\
& \left[\quad \sin^2(x) = \frac{1}{2}\big(1 - \cos(2x)\big) \quad \right] \\
=& \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty} p_n \cos(2tg\sqrt{n}), \qquad p_n = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}.
\end{aligned} \tag{4}$$

If one evaluates $P_+(t)$ (see figure 1), one sees a damped oscillation. To get a qualitative understanding of this, we note that $P_+(t)$ is a sum of oscillatory functions with different frequencies. For sufficiently small t, all the significant terms are in phase, meaning that they interfere constructively, i.e., in total we get a large amplitude oscillation. However, as t increases, the different angular speeds are getting more noticed, such that the different terms fall more and more out of phase, with the consequence that the constructive interference gradually turns destructive, and we see a damping of the oscillatory motion (often referred to as "dephasing"). In the following, we would like to get a quantitative understand of the frequency of the oscillation, and of the rate of the damping. In order to obtain this, we make similar approximations as we did in the previous section, i.e., we will replace the sum by an integral, and the Poisson distribution with the normal distribution. Hence, we here aim at an understanding the behavior of P_+ for large values of $|\alpha|^2$. We approximate the sum in (4) as

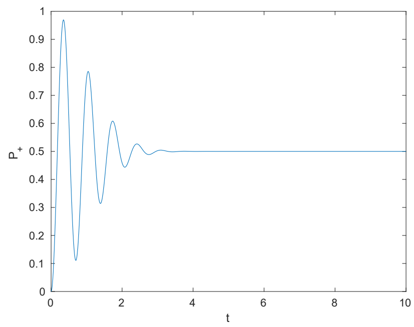


Figure 1: Explicit evaluation of $P_+(t)$ in (4) for $|\alpha|^2 = 20$.

$$
\begin{aligned}
&\sum_{n=1}^{\infty} p_n \cos(2tg\sqrt{n}) \\
\approx &\int f_{0,1}(x)\cos\left(2tg\sqrt{|\alpha|^2 + x|\alpha|}\right)dx \\
= &\int f_{0,1}(x)\cos\left(2tg|\alpha|\sqrt{1+\frac{x}{|\alpha|}}\right)dx \\
&[\quad \text{Taylor expansion } \sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2) \quad] \\
\approx &\int f_{0,1}(x)\cos\left(2tg|\alpha|\big(1+\frac{x}{2|\alpha|}\big)\right)dx \\
= &\int f_{0,1}(x)\cos\left(t\big(2g|\alpha| + xg\big)\right)dx \\
&[\quad y = 2g|\alpha| + xg \quad] \\
= &\int f_{2g|\alpha|,g}(y)\cos(ty)dy \\
= &\mathrm{Re}\int f_{2tg|\alpha|,tg}(y)e^{ity}dy \\
&[\quad \text{Fourier transform (the characteristic function) of the} \\
&\quad \text{normal distribution is } e^{i\mu t}e^{-t^2\sigma^2/2}. \quad] \\
= &\mathrm{Re}\ e^{i2g|\alpha|t}e^{-t^2g^2/2} \\
= &\cos(2g|\alpha|t)e^{-t^2g^2/2}.
\end{aligned}
$$

Hence

$$
P_+^{\mathrm{approx}}(t) = \frac{1}{2} - \frac{1}{2}\cos(2g|\alpha|t)e^{-t^2g^2/2}. \tag{5}
$$

The result is thus a damped oscillatory motion with angular frequency $2g|\alpha|$. One may also note that the decay is not exponential, but rather of the form e^{-t^2}. With these approximations it is rather unclear how close $P_+^{\mathrm{approx}}(t)$ actually is to $P_+(t)$. However, a numerical evaluation (see figure 2 (a)) suggests that the approximation is very good. Or at least it is good for sufficiently small times. If one looks at larger times, $P_+(t)$ very suddenly undergoes revivals (see figure 2 (b)). In our approximate evaluation of $P_+^{\mathrm{approx}}(t)$, we made a continuum approximation, while the actual sum in (4) is discreet. The source of these revivals is the discreetness of the frequencies of the oscillatory components.

To see how this can come about, let us consider the much simpler case of a sum of only two oscillatory functions $\cos(t\omega_1) + \cos(t\omega_2)$. The amplitude of the sum is large when $t\omega_1$ and $t\omega_2$ are in phase, i.e., when $|t\omega_1 - t\omega_2| = n2\pi$ for some $n \in \mathbb{Z}$. In other words, the sum will show constructive interference around the times

$$t_n := n\frac{2\pi}{|\omega_1 - \omega_2|}, \tag{6}$$

where the first revival happens for $n = 1$ at $t_1 = 2\pi/|\omega_2 - \omega_1|$. Hence, the closer the two frequencies are, the longer before the first revival occurs.

From our previous discussions, we know that the photon number distribution is is peaked near $n_{\mathrm{peak}} \approx |\alpha|^2$. Hence, the frequency that yields the dominating contribution to the sum (4) in should be located approximately at $\omega_1 = 2g\sqrt{n_{\mathrm{peak}}} \approx 2g\sqrt{|\alpha|^2}$. The photon numbers closest to the dominating peak are at $n_{\text{next to peak}} = n_{\mathrm{peak}} \pm 1$. Hence, the closest frequencies to ω_1 should be at $\omega_2 = 2g\sqrt{n_{\mathrm{peak}} \pm 1} \approx 2g\sqrt{|\alpha|^2 \pm 1}$. If we now make the wild hypothesis that the dominating frequency, and one of the next to dominating frequencies, are the main factors for determining the first revival, we would insert ω_1 and ω_2 in (6) to obtain

$$\begin{aligned} t_n =& \frac{n2\pi}{|\omega_2 - \omega_1|} \\ =& \frac{n2\pi}{\left|2g\sqrt{|\alpha|^2 \pm 1} - 2g\sqrt{|\alpha|^2}\right|} \\ =& \frac{n2\pi}{2g|\alpha|}\frac{1}{\left|\sqrt{1 \pm \frac{1}{|\alpha|^2}} - 1\right|} \\ \approx& \frac{n2\pi}{2g|\alpha|}\frac{1}{\left|1 \pm \frac{1}{2|\alpha|^2} - 1\right|} \\ \approx& n\frac{2\pi|\alpha|}{g}. \end{aligned} \tag{7}$$

It is far from clear that this reasoning produces anything relevant, but we can compare this result with an explicit evaluation of $P_+(t)$. In figure 2(b) the values t_n are indicated as green vertical lines. At least for the first few revivals, there is a good agreement.

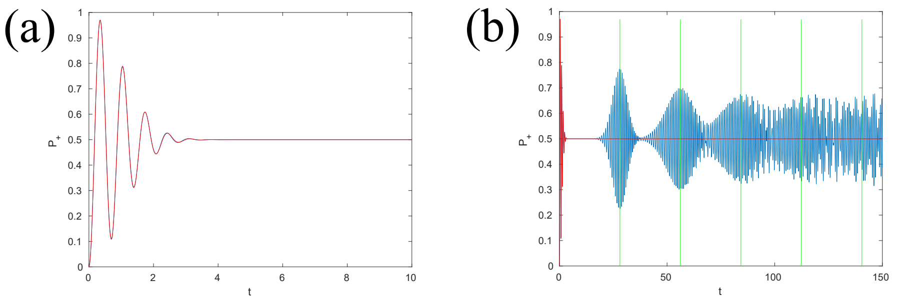


Figure 2: (a) For $|\alpha|^2 = 20$ we here plot $P_+(t)$ (blue line) and the approximation (5) (red line). For this small scale, the difference is not really visible.
(b) For larger times, $P_+(t)$ undergo revivals. The green vertical lines are positioned at the times t_n, determined in (7), which give crude estimates for when the revivals occur.