Identical particles

Lecture October 15, 2019

So far we have said that when we combine two, or more, systems into a larger system, we should take the tensor product of their Hilbert spaces. However, empirically it seems like this is not a good idea when we are dealing with identical particles. It seems that the tensor-product space is "too big".

The problem is that when we assign the Hilbert space $\mathcal{H}\otimes\mathcal{H}$ to two particles, we allow for observables such as

$$X_1 \otimes \hat{1}_2, \quad \hat{1}_1 \otimes X_2, \tag{1}$$

which would correspond to queries like, "where is particle 1", or "where is particle 2". The question is, how would we construct experiments that would implement these questions? If the particles are indistinguishable, we cannot construct a device that would single out one of them.

This observation is not unique for quantum mechanics. In classical mechanics we can also have identical particles, i.e., particles for which all properties are the same (e.g., mass and charge). Consider for example two particles on a line. We can describe the configuration of these two particles via the vector (x_1, x_2). However, we could equally well describe it as (x_2, x_1); if all properties of the two particles are the same, this would make no difference. There are various ways in which we could remove this ambiguity, e.g., by demanding that we only use coordinates (x_1, x_2) such that $x_1 \leq x_2$. However, as far as we know, this ambiguity has never caused any problem in classical mechanics, so we can equally well let it be.[1]

However, if we turn to quantum mechanics, not dealing with these issues leads to a theory that does not fit particularly well with empirical observations. For example, our theory would only correctly describe atoms with at most one electron. There are potentially different ways to deal with this, and ultimately it is an empirical question. A successful approach has been to start with the full tensor product Hilbert space (as if the particles were distinguishable), and then restrict to states that are symmetric under imagined permutations of the particles. This construction gives rise to fermions and bosons, which as far as we know, account for all the elementary particles.

1 The group of permutations

Since our whole construction rests on the notion of permutations of particles, it is useful to consider some of the mathematical properties of permutations.

[1] Removing the redundant could potentially even cause some headache. For example, the rule that we should keep $x_1 \leq x_2$, could be somewhat awkward to combine with the differential equations of Newtonian mechanics.

There is quite a lot to say about permutations, but we only need a few details.

A permutation π of n letters is a way of re-arranging the symbols $1, 2, \dots, n$, i.e., π is a bijection of the set $\{1, \dots, n\}$. There are various ways of denoting permutations. One example is the two-line notation, where the first line contains the numbers $1, \dots, n$ and the i-th palace of the second line is $\pi(i)$

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix},$$

with a concrete example being

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

One can think of this in terms of a graph where arrows point to where each letter is mapped (see Figure 1(a)). One can multiply permutations in the sense that if we apply one permutation after the other (see Figure 1(b)), then we get a new permutation. The set of permutations on n letters forms a group, denoted S_n, and referred to as the *symmetric group*.

How many permutations of n letters are there? One can think of in how many ways that the lower row in the two-line notation can be filled. We can fill the first slot in n ways, the second in $n-1$ ways, etc, with the result being

$$|S_n| = n(n-1)\cdots 2 \cdot 1 = n! \approx e^{n \ln n}$$

(this is a lot!)

A transposition is a special permutation where only two letters are permuted. For example, the following is a transposition

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}.$$

We will make use of the following facts (that we will not prove):

- Every permutation can be written as a sequence of transpositions.
- For a given permutation, the number of transpositions is either even for all decompositions, or odd for all decompositions.

Hence, the set of permutations S_n decomposes into two subsets; one with even decompositions and one with odd. This justifies the following definition

$$\operatorname{sgn}(\sigma) = \begin{cases} + & \text{even} \\ - & \text{odd} \end{cases}$$

2 Identical particles

Choose some single-particle Hilbert space $\mathcal{H}^{(1)}$ and let

$$\mathcal{H}^{(n)} = \mathcal{H}^{(1)} \otimes \cdots \otimes \mathcal{H}^{(1)}$$

be a tensor product of n copies of $\mathcal{H}^{(1)}$. If we choose some basis $\{|i\rangle\}_i$ of the single-particle space $\mathcal{H}^{(1)}$, then

$$\{|i_1, i_2, \dots, i_n\rangle\}_{i_1,\dots,i_n}$$

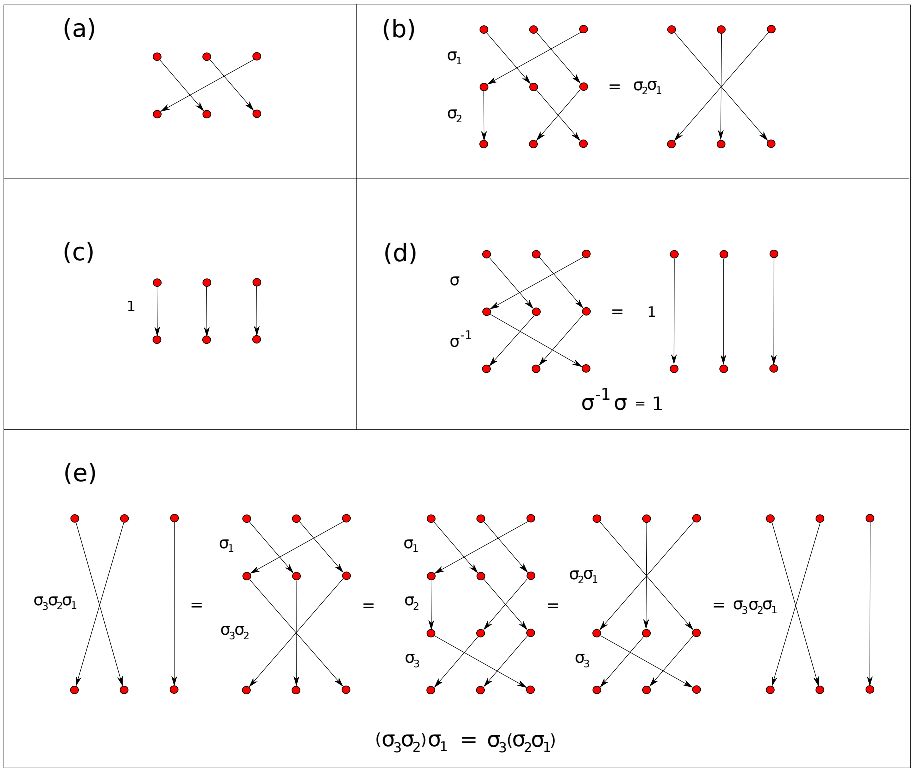


Figure 1: (a) Each permutation can be regarded as a graph, where each position indicates a letter, and where the arrows points to where each letter is mapped.
(b) One can multiply permutations σ_1 and σ_2 by performing one after the other, resulting in the new permutation $\sigma_2\sigma_1$. We obtain the arrows in the right graph, by following the arrows in the left.
By observing some examples, one can make it plausible that S_n is a group. First of all it seems reasonable that all compositions of permutations (as in (b)) is again a permutation. (c) There is an identity, which simply leaves everything as it is. This has the desired effect that $1\sigma = \sigma 1$. (d) For every permutation there is an inverse that 'undo' it, i.e., $\sigma^{-1}\sigma = \sigma\sigma^{-1} = 1$. (e) The product is associative $\sigma_3(\sigma_2\sigma_1) = (\sigma_3\sigma_2)\sigma_1$.

is a basis of $\mathcal{H}^{(n)}$.

For an element $\pi \in S_n$ we define its action on $\mathcal{H}^{(n)}$ by first defining how it acts on the basis elements.

$$\pi : |i_1, \ldots, i_n\rangle \mapsto |\pi(i_1), \ldots, \pi(i_n)\rangle$$

This is extended linearly to the the whole of $\mathcal{H}^{(n)}$. This means that if we have an arbitrary element of $\mathcal{H}^{(n)}$, then we can expand it in the basis as

$$|\psi\rangle = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} |i_1, \ldots, i_n\rangle,$$

and we define $\pi|\psi\rangle$ as

$$\begin{aligned}\pi|\psi\rangle &= \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n}\pi|i_1,\ldots,i_n\rangle \\ &= \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n}|\pi(i_1),\ldots,\pi(i_n)\rangle.\end{aligned}$$

Hence, we have defined the action of π on the whole of $\mathcal{H}^{(n)}$.

3 Bosons

A vector $|\psi\rangle \in \mathcal{H}^{(n)}$ is called *totally symmetric* if

$$\pi|\psi\rangle = |\psi\rangle, \quad \text{for all} \quad \pi \in S_n.$$

In words, this means that if a state remains the same, no matter how we shuffle around the particles, then it is totally symmetric.

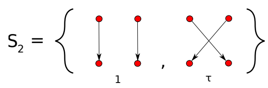


Figure 2: S_2 only consists of two elements, the identity 1, and the transposition τ that swaps 1 and 2.

Example: Recall that the triplet states are particular spin-states of two spin-half particles. In this case, the single-particle space is two-dimensional $\mathcal{H}^{(1)} = \text{Sp}\{|+\rangle, |-\rangle\}$. We have two particles, so $n = 2$, and S_2 only consists of two elements, namely the identity-permutation 1 (that leaves everything as it is), and the permutation τ that swaps the two particles. In other words $S_2 = \{1, \tau\}$. The three triplet states

$$|+,+\rangle, \quad \frac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle), \quad |-,-\rangle \tag{2}$$

are examples of totally symmetric vectors. Let us confirm this for the triplet state $\frac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle)$. We thus need to show that $\pi\frac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle) = \frac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle)$, for all $\pi \in S_2$. It is trivially true that the identity permutation 1 leaves $\frac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle)$. Hence, it remains to show that τ also leaves $\frac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle)$ invariant

$$\begin{aligned}\tau\frac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle) =& \frac{1}{\sqrt{2}}(\tau|+,-\rangle + \tau|-,+\rangle) \\ =& \frac{1}{\sqrt{2}}(|-,+\rangle + |+,-\rangle) \\ =& \frac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle).\end{aligned} \tag{3}$$

Hence, we have shown that this vector is totally symmetric.

The set of totally symmetric vectors form a linear subspace. To see this, let $|\psi\rangle, |\psi'\rangle \in \mathcal{H}^{(n)}$ be two totally symmetric vectors, and let $\pi \in S_n$. Then

$$\begin{aligned}\pi(\alpha|\psi\rangle + \beta|\psi'\rangle) =& \alpha\pi|\psi\rangle + \beta\pi|\psi'\rangle \\ & \qquad [\text{Since } |\psi\rangle \text{ and } |\psi'\rangle \text{ are totally symmetric}] \\ =& \alpha|\psi\rangle + \beta|\psi'\rangle.\end{aligned} \tag{4}$$

Hence, any linear combination of totally symmetric vectors, is again totally symmetric. We can thus conclude that the set of totally symmetric vectors forms a vector space. We denote this vector space by

$$\mathrm{Sym}^{(n)}(\mathcal{H}^{(1)}) = \Big\{ |\psi\rangle \in \mathcal{H}^{(n)} : \pi|\psi\rangle = |\psi\rangle, \quad \forall \pi \in S_n \Big\} .$$

We make the following definition:

- Particles whose n-body Hilbert space is $\mathrm{Sym}^{(n)}(\mathcal{H}^1)$, are called bosons.

At this point of our theory, it is strictly speaking unclear whether bosons exist, but they do, e.g., photons.

In the following we take a closer look on the space $\mathrm{Sym}^{(n)}(\mathcal{H}^1)$. We define the *symmetrization operator* on $\mathcal{H}^{(n)}$ as

$$P_S : |\psi\rangle \mapsto \frac{1}{n!} \sum_{\pi \in S_n} \pi|\psi\rangle$$

The following is true:

- P_S is the (orthogonal) projector onto $\mathrm{Sym}^{(n)}(\mathcal{H}^1)$.

In the lecture we did not prove this, but if you are interested, you can find a proof in Appendix A below. Here, we only prove that P_S leaves every element in $\mathrm{Sym}^{(n)}(\mathcal{H}^1)$ intact. To see this, suppose that $|\psi\rangle \in \mathrm{Sym}^{(n)}(\mathcal{H}^1)$ then

$$\begin{aligned} P_S|\psi\rangle =& \frac{1}{n!} \sum_{\pi \in S_n} \pi|\psi\rangle \\ & [\text{By assumption } \pi|\psi\rangle = |\psi\rangle] \\ =& \frac{1}{n!} \sum_{\pi \in S_n} |\psi\rangle \\ & [\quad |S_n| = n! \quad] \\ =& |\psi\rangle. \end{aligned} \tag{5}$$

Hence, every element of $\mathrm{Sym}^{(n)}(\mathcal{H}^1)$ is left invariant, and consequently $P_S\mathrm{Sym}^{(n)}(\mathcal{H}^1) = \mathrm{Sym}^{(n)}(\mathcal{H}^1)$.

Since P_S is the (orthogonal) projector onto $\mathrm{Sym}^{(n)}(\mathcal{H}^1)$, it means that we can use P_S in order to find a set of vectors that spans $\mathrm{Sym}^{(n)}(\mathcal{H}^1)$. For example, with $\{|i_1, \ldots, i_n\rangle\}_{i_1,\ldots,i_n}$ being a basis of the n-particel space $\mathcal{H}^{(n)}$, it follows that $\{P_S|i_1, \ldots, i_n\rangle\}_{i_1,\ldots,i_n}$ spans $\mathrm{Sym}^{(n)}(\mathcal{H}^1)$.

Example: $\{|+,+\rangle, |+,-\rangle, |-,+\rangle, |-,-\rangle\}$ is a basis of the Hilbert space of two spin-half particles. we recall that $S_2 = \{1, \tau\}$, where 1 is the identity permutation, and τ is the transposition of the two particles.

$$\begin{aligned} P_S|+,+\rangle =& \frac{1}{n!} \sum_{\pi \in S_n} \pi|+,+\rangle \\ =& \frac{1}{2}(1|+,+\rangle + \tau|+,+\rangle) \\ =& \frac{1}{2}(|+,+\rangle + |+,+\rangle) \\ =& |+,+\rangle. \end{aligned} \tag{6}$$

Analogously

$$
\begin{aligned}
P_S|+,-\rangle =&\frac{1}{2}(1|+,-\rangle + \tau|+,-\rangle) \\
=&\frac{1}{2}(|+,-\rangle + |-,+\rangle)
\end{aligned}
$$

$$
\begin{aligned}
P_S|-,+\rangle =&\frac{1}{2}(1|-,+\rangle + \tau|-,+\rangle) \\
=&\frac{1}{2}(|-,+\rangle + |+,-\rangle) \\
=&\frac{1}{2}(|+,-\rangle + |-,+\rangle),
\end{aligned} \tag{7}
$$

$$
P_S|-,-\rangle =|-,-\rangle.
$$

Hence the three vectors $|+,+\rangle$, $\frac{1}{2}(|+,-\rangle + |-,+\rangle)$, and $|-,-\rangle$ are all totally symmetric.

We can make some observations. First, that the resulting vectors are not necessarily normalized. Second, one can see that the resulting vector only depends on the number of times that a given single-particle basis element occurs in the initial vector, not the order that they occur (e.g. $|+,-\rangle$ and $|-,+\rangle$ are both mapped to the same vector $\frac{1}{2}(|+,-\rangle + |-,+\rangle)$). This is an observation that we will come back to in some later part of these lectures.

4 Fermions

A vector $|\psi\rangle \in \mathcal{H}^{(n)}$ is said to be *totally anti-symmetric* if

$$
\pi|\psi\rangle = \text{sgn}(\sigma)|\psi\rangle, \quad \text{for all} \quad \pi \in S_n
$$

Example: We again consider or favorite example of two spin-half particles, where we only have two permutations: the identity 1 and the transposition τ. The identity of course leaves the singlet state

$$
\frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle)
$$

invariant, but for the transposition τ we have

$$
\begin{aligned}
\tau\frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle) =&\frac{1}{\sqrt{2}}(\tau|+,-\rangle - \tau|-,+\rangle) \\
=&\frac{1}{\sqrt{2}}(|-,+\rangle - \tau|+,-\rangle) \\
=&-\frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle).
\end{aligned}
$$

Since, τ is a single transposition, it follows that

$$
\text{sgn}(\tau) = 1. \tag{8}
$$

Hence, we can conclude that the singlet state is a totally anti-symmetric vector.

Analogously to the totally symmetric space, we can ask whether the set of totally anti-symmetric vectors form a vector space. We can use precisely the same argument as for the totally symmetric case. Let us assume that $|\psi\rangle$ and $|\psi'\rangle$ are totally anti-symmetric then

$$\begin{aligned}\pi(\alpha|\psi\rangle + \beta|\psi'\rangle) =& \alpha\pi|\psi\rangle + \beta\pi|\psi'\rangle \\ =& \alpha\text{sgn}(\sigma)|\psi\rangle + \beta\text{sgn}(\sigma)|\psi'\rangle \\ =& \text{sgn}(\sigma)[\alpha|\psi\rangle + \beta|\psi'\rangle]\end{aligned} \tag{9}$$

Hence, a linear combination of totally anti-symmetric vectors is again totally anti-symmetric. Hence, they form a vector space. We let

$$\wedge^n(\mathcal{H}^{(1)}) = \Big\{|\psi\rangle \in \mathcal{H}^{(n)} : \pi|\psi\rangle = \text{sgn}(\pi)|\psi\rangle\Big\}$$

denote the space of totally anti-symmetric vectors in $\mathcal{H}^{(n)}$ (where the symbol "$\wedge^n$" is pronounced "wedge-n"). We make the following definition:

- Particles whose n-particle Hilbert space is $\wedge^n(\mathcal{H}^1)$ are called fermions.

Electrons seems to be fermions in this sense.

We define the *anti-symmetrization* operator as

$$P_A : |\psi\rangle \mapsto \frac{1}{n!}\sum_{\pi\in S_n} \text{sgn}(\pi)\pi|\psi\rangle.$$

The anti-symmetrization operator it the projector onto the totally anti-symmetric space $\wedge^n(\mathcal{H}^{(1)})$ in the multi-particle space $\mathcal{H}^{(n)}$. In the lecture we did not prove this, but only checked that all elements of $\wedge^n(\mathcal{H}^{(1)})$ are left invariant by P_A. So, assume $|\psi\rangle \in \wedge^n(\mathcal{H}^1)$, then

$$\begin{aligned}P_A|\psi\rangle =& \frac{1}{n!}\sum_{\pi\in S_n} \text{sgn}(\pi)\pi|\psi\rangle \\ =& \frac{1}{n!}\sum_{\pi\in S_n} \text{sgn}(\pi)\text{sgn}(\pi)|\psi\rangle \\ =& \frac{1}{n!}\sum_{\pi\in S_n} |\psi\rangle \\ =& |\psi\rangle.\end{aligned}$$

Hence, every element of $\wedge^n(\mathcal{H}^1)$ is left invariant, and consequently $P_A\wedge^n(\mathcal{H}^1) = \wedge^n(\mathcal{H}^1)$.

Analogous to the totally symmetric case, $\wedge^n(\mathcal{H}^1)$ is spanned by $\{P_A|i_1,\ldots,i_n\rangle\}_{i_1,\ldots,i_n}$.

Example: $\{|+,+\rangle, |+,-\rangle, |-,+\rangle, |-,-\rangle\}$ is a basis of the tensor product space of two spin-half particles.

$$\begin{aligned}P_A|+,+\rangle =& \frac{1}{n!}\sum_{\pi\in S_n} \text{sgn}(\pi)\pi|+,+\rangle = \frac{1}{2}(|+,+\rangle - |+,+\rangle) = 0, \\ P_A|+,-\rangle =& \frac{1}{2}(|+,-\rangle - |-,+\rangle), \\ P_A|-,+\rangle =& \frac{1}{2}(|-,+\rangle - |+,-\rangle) = -\frac{1}{2}(|+,-\rangle - |-,+\rangle), \\ P_A|-,-\rangle =& 0.\end{aligned}$$

In this example one should in particular note $P_A|+,+\rangle = 0$ and $P_A|-,-\rangle = 0$. This is a particular case of a simple, but very important observation concerning fermions, namely that no two fermions can occupy the same single-particle state. Let us investigate this a bit further. Suppose that a basis element $|i_1,\ldots,i_n\rangle$ is such that two particles, particle j and k (with $j \neq k$) say, occupy the same single-particle state. In other words

$$i_j = i_k.$$

If particles j and k are in the same state, it means that if we swap them, then the state remains the same. In other words, if τ_{jk} is the transposition that swaps particles j and k, then

$$\tau_{jk}|i_1,\ldots,i_n\rangle = |i_1,\ldots,i_n\rangle. \tag{10}$$

From this we can see that $|i_1,\ldots,i_n\rangle$ cannot be an element of $\wedge^n(\mathcal{H}^1)$, since if it was, we would have

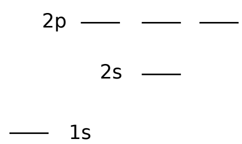
$$\tau_{jk}|i_1,\ldots,i_n\rangle = \mathrm{sgn}(\tau_{jk})|i_1,\ldots,i_n\rangle = -|i_1,\ldots,i_n\rangle,$$

which is in contradiction with (10). In Appendix B we show a stronger statement, namely that $P_A|i_1,\ldots,i_n\rangle = 0$ if (10) holds. This means that $|i_1,\ldots,i_n\rangle$ is orthogonal to $\wedge^n(\mathcal{H}^1)$, whenever two particles occupy the same single particle state.

What we have discovered is nothing but the Pauli-principle. As you may recall, the Pauli-principle has very far reaching consequences. To see why, let the single particle basis-elements $|i\rangle$ be the energy-eigenvectors of a single-particle Hamiltonian, with corresponding energy eigenvalues $E_0, E_1, \ldots$, where $E_0 < E_1 < E_2 < \cdots$. Consider the case that we would have a collection of n *distinguishable* particles that do not interact with each other, but all having the same single-particle Hamiltonian. Now we can ask, what is the ground state of this system? Clearly, every particle independently finds its ground state, such that the total ground state energy is nE_0. Let us now instead imagine that the n particles are identical fermions, then what is the ground state energy? Now, at most one particle can occupy the ground state. The lowest energy that the next particle can have is E_1, etc. Hence, the ground state is $\sum_{k=0}^{n-1} E_k$.

5 A crude sketch of the periodic table

As an application of what we just have found, we consider a crude model of atoms, which yields a sketch of the periodic table (or at least the beginning of it).

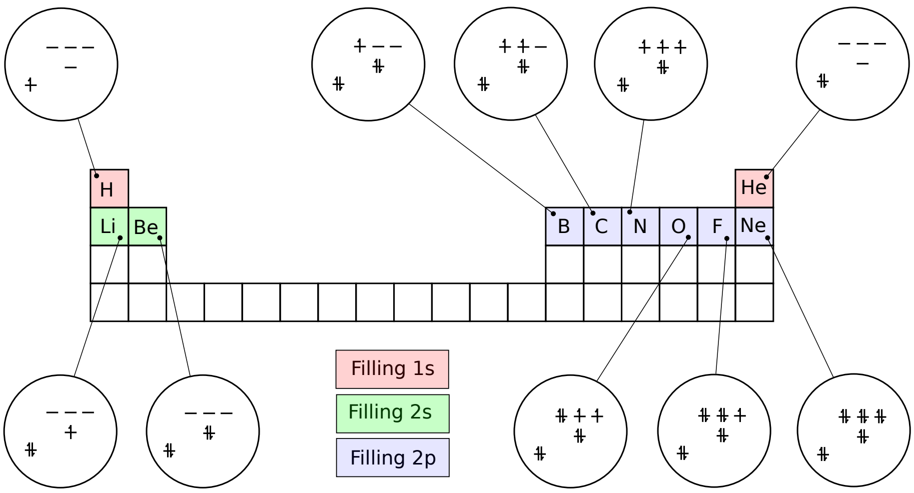


The eigenvectors of a single particle in a central potential are of the form

$$\psi_{nlm}(r,\theta,\varphi) = \phi_{nl}(r)Y_l^m(\theta,\varphi), \tag{11}$$

where $\phi_{nl}(r)$ are radial wave-functions, and Y_l^m are the spherical harmonics (angular wave-functions), where $l = 0, 1, \ldots$ is the angular quantum number, and $m = -l, \ldots, l$ is the magnetic quantum number, and n the principal quantum number. By tradition, the $l = 0$ orbitals are referred to a s-orbitals, and $l = 1$ as p-orbitals. One moreover tends to put the principal quantum number before the orbital, e.g., $1s$ stands for $n = 0, l = 0$. One should keep in mind that the electron also has a spin-degree of freedom since it is a spin-half fermion. Hence, for each spatial state there are two spin-states ($|+\rangle$ and $|-\rangle$).

In our model, we assume that the electrons do not interact with each other (although they of course in actual fact do interact), and find the ground state as we add more and more electrons. For the hydrogen atom, all orbitals with the same principal quantum number have the same energy. However, for atoms with a higher number of electrons, there is a splitting, where lower angular momentum has lower energy, so the $2s$ orbital has a lower energy than the $2p$ orbitals. Another question is in what manner the degenerate p-orbitals should be filled. The non-interacting model does not really answer this. The rule turns out to be that one should fill the unoccupied orbitals before one starts to occupy them doubly. Since we in addition have two spin-states, we can fill the $1s$-orbital with two electrons; the first yielding the hydrogen atom, and the second yielding helium. Next, we fill up the $2s$-orbital, which results in lithium and beryllium. Next, the $2p$-orbitals are filled, giving boron, carbon, nitrogen, oxygen, fluorine, and neon. We can conclude that, with a simple non-interacting particle-model, the Pauli-principle, and some additional rules put in by hand (i.e., cheating), we can get a sketch of the periodic table



A Appendix: P_S is the (orthogonal) projector onto $\mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$

In the lecture we did not prove that P_S is the (orthogonal) projector onto $\mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$. For the sake of completeness, it is proved here for those who are interested.

We start with a couple of general observations that are true for any finite group, not only the symmetric group. Suppose that we have a (for the sake of simplicity) finite group G. If $g \in G$, then the mapping $x \mapsto xg$ is a bijection on G. A more compact way of saying the same thing is that $Gg = G$. In words, this means that the if we multiply each element in the set G from the right with g, then we regain the set G.

In order to show this, we need to show that the mapping $x \mapsto xg$ is both injective and surjective. For injectivity, suppose that we have $x, y \in G$ such that $xg = yg$. However, since there exists an inverse to g, it is the case that

$$\begin{aligned} xg =& yg \\ &\Rightarrow \\ xgg^{-1} =& ygg^{-1} \\ &\Rightarrow \\ x1 =& y1 \\ &\Rightarrow \\ x =& y \end{aligned} \tag{12}$$

Hence, if two elements maps to the same element, then the two initial elements have to be the same. Hence, $x \mapsto xg$ is injective. Next, we wish to show that it is surjective. Thus take any $y \in G$, then we note that $x := yg^{-1} \in G$, and thus $xg = yg^{-1}g = y1 = y$. Hence, for every element y in G we can find an element x in G such that $x \mapsto xg$ yields y. Hence $x \mapsto xg$ is surjective.

In an analogous manner one can also show that the mapping $x \mapsto gx$ is a bijection, and thus $gG = G$. In a very similar spirit, one can also show that the mapping $x \mapsto x^{-1}$ is a bijection on G, which we can write more succinctly as $G^{-1} = G$. Hence, if we take the inverse of every element in the set G, then we again obtain the set G. To see that this is true, we first show that $x \mapsto x^{-1}$ is injective

$$\begin{aligned} x^{-1} =& y^{-1} \\ &\Rightarrow \\ x^{-1}x =& y^{-1}x \\ &\Rightarrow \\ 1 =& y^{-1}x \\ &\Rightarrow \\ y1 =& yy^{-1}x \\ &\Rightarrow \\ y =& 1x \\ &\Rightarrow \\ y =& x. \end{aligned} \tag{13}$$

Hence, $x \mapsto x^{-1}$ is injective. Moreover, if $y \in G$, then $x := y^{-1} \in G$ and $x^{-1} = (y^{-1})^{-1} = y$. Hence, $x \mapsto x^{-1}$ is surjective.

A particular finite group is the set of permutations over n letters, S_n, and thus according to the discussion above, we have

$$S_n\sigma = S_n, \quad \forall \sigma \in S_n, \tag{14}$$

$$\sigma S_n = S_n, \quad \forall \sigma \in S_n, \tag{15}$$

and

$$S_n^{-1} = S_n. \tag{16}$$

We shall use these observations in order to prove the following.

Proposition 1.
P_S is the (orthogonal) projector onto $Sym^{(n)}(\mathcal{H}^{(1)})$ in $\mathcal{H}^{(n)}$.

Proof. We first prove that P_S is an (orthogonal) projector

$$\begin{aligned} P_S^\dagger =& \frac{1}{n!} \sum_{\pi \in S_n} \pi^\dagger \\ =& \frac{1}{n!} \sum_{\pi \in S_n} \pi^{-1} \\ & [\quad \pi' = \pi^{-1} \quad] \\ =& \frac{1}{n!} \sum_{\pi' \in S_n^{-1}} \pi' \\ & [\quad S_n^{-1} = S_n \quad] \\ =& \frac{1}{n!} \sum_{\pi' \in S_n} \pi' \\ =& P_S. \end{aligned} \tag{17}$$

Next we wish to show that $P_S^2 = P_S$, but first we show the following

$$\begin{aligned} P_S \sigma =& \frac{1}{n!} \sum_{\pi \in S_n} \pi\sigma \\ & [\quad \pi' = \pi\sigma \quad] \\ =& \frac{1}{n!} \sum_{\pi' \in S_n \sigma} \pi' \\ & [\quad S_n \sigma = S_n \quad] \\ =& \frac{1}{n!} \sum_{\pi' \in S_n} \pi' \\ =& P_S. \end{aligned} \tag{18}$$

It follows that

$$\begin{aligned} P_S^2 =& P_S \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \\ =& \frac{1}{n!} \sum_{\sigma \in S_n} P_S \sigma \\ & [\text{By } (18)] \\ =& \frac{1}{n!} \sum_{\sigma \in S_n} P_S \\ & [\quad |S_n| = n! \quad] \\ =& P_S. \end{aligned} \tag{19}$$

Hence, we can conclude that $P_S^\dagger = P_S$ and $P_S^2 = P_S$ and thus P_S is an (orthogonal) projection.

Next we wish to show that P_S is the projector onto $\mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$ in $\mathcal{H}^{(n)}$. First, we will show is that for any vector $|\psi\rangle \in \mathcal{H}^{(n)}$ it is the case that $P_S|\psi\rangle \in \mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$, i.e., that $P_S\mathrm{Sym}^{(n)}(\mathcal{H}^{(1)}) \subseteq \mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$. Thus, assume that $|\psi\rangle \in \mathcal{H}^{(n)}$. Then it is the case that

$$\begin{aligned}\sigma P_S|\psi\rangle =& \frac{1}{n!}\sum_{\pi\in S_n}\sigma\pi|\psi\rangle \\ &[\quad \pi' = \sigma\pi \quad] \\ =&\frac{1}{n!}\sum_{\pi'\in\sigma S_n}\pi'|\psi\rangle \\ &[\quad \sigma S_n = S_n \quad] \\ =&\frac{1}{n!}\sum_{\pi'\in S_n}\pi'|\psi\rangle \\ =&P_S|\psi\rangle.\end{aligned} \tag{20}$$

Hence, we can conclude that $\sigma P_S|\psi\rangle = P_S|\psi\rangle$ for all $\sigma \in S_n$. Hence, by definition $P_S|\psi\rangle \in \mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$. Hence, we can conclude that $P_S\mathcal{H}^{(n)} \subseteq \mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$. Finally, we need to show that we reach *all* of $\mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$, i.e., that $P_S\mathcal{H}^{(n)} = \mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$. However, in equation (5) in the main text, we showed that $P_S|\psi\rangle = |\psi\rangle$ for all $\mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$. Hence, $P_S\mathrm{Sym}^{(n)}(\mathcal{H}^{(1)}) = \mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$. By combining this with $P_S\mathcal{H}^{(n)} \subseteq \mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$, we can conclude that $P_S\mathcal{H}^{(n)} = \mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$. Hence, P_S is the (orthogonal) projector onto $P_S|\psi\rangle \in \mathrm{Sym}^{(n)}(\mathcal{H}^{(1)})$. □

B Appendix: $P_A|i_1,\ldots,1_n\rangle = 0$ if $i_j = i_k$ for some $k \neq j$

In the lecture we proved that if $|i_1,\ldots,1_n\rangle$ is such that $i_j = i_k$ for some $k \neq j$, then $|i_1,\ldots,1_n\rangle$ cannot be an element in $\wedge^n(\mathcal{H}^1)$. Here we prove the stronger statement that $P_A|i_1,\ldots,1_n\rangle = 0$, which means that $|i_1,\ldots,1_n\rangle$ is orthogonal to $\wedge^n(\mathcal{H}^1)$.

We shall prove a more general statement, namely:

Proposition 2. *If $|\psi\rangle \in \mathcal{H}^{(n)}$ is such that $\tau|\psi\rangle = |\psi\rangle$ for some transposition $\tau \in S_n$, then $P_A|\psi\rangle = 0$.*

Proof. First we note the following, τ be any transposition in S_n, then

$$\begin{aligned}
P_A\tau =& \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\sigma\tau \\
=& \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma\tau\tau^{-1})\sigma\tau \\
& [\quad \mathrm{sgn}(ab) = \mathrm{sgn}(a)\mathrm{sgn}(b) \quad] \\
=& \mathrm{sgn}(\tau^{-1})\frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma\tau)\sigma\tau \\
& [\quad \sigma' = \sigma\tau \quad] \\
=& \mathrm{sgn}(\tau^{-1})\frac{1}{n!}\sum_{\sigma'\in \tau S_n} \mathrm{sgn}(\sigma')\sigma' \\
& [\quad \tau S_n = S_n \quad] \\
=& \mathrm{sgn}(\tau^{-1})\frac{1}{n!}\sum_{\sigma'\in S_n} \mathrm{sgn}(\sigma')\sigma' \\
=& \mathrm{sgn}(\tau^{-1})P_A \\
=& - P_A.
\end{aligned}$$

Now we assume that $|\psi\rangle$ is such that $\tau|\psi\rangle = |\psi\rangle$, then

$$P_A|\psi\rangle = P_A\tau|\psi\rangle = -P_A|\psi\rangle.$$

Thus we can conclude that $P_A|\psi\rangle = 0$. □

As we already have remarked in the main text, if $|i_1, \dots, 1_n\rangle$ is such that $i_j = i_k$ for some $k \neq j$, then it means that the transposition τ_{jk} between particle j and k is such that $\tau_{jk}|i_1, \dots, 1_n\rangle = |i_1, \dots, 1_n\rangle$. Hence, by Proposition 2 we can conclude that $P_A|i_1, \dots, 1_n\rangle = 0$.