Advanced Quantum Mechanics

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Exercise sheet 1 (Due: Monday October, 14th.)

1.1 Linear operators do not generally commute

Consider the two matrices

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Calculate ZX and XZ. Is ZX equal to XZ? **(1 point)**

Moral of this exercise:

Do not swap the order of linear operators unless you can justify it!

If you keep this in mind for the rest of this course[1], then you (and your tutors[2]) will have a much easier life.

1.2 Useful relations

We will often have to work with expressions that involve commutation, or functions of operators. Here we show some relations that can be quite useful. In the following $[A, B] = AB - BA$ is the commutator, and $\{A, B\} = AB + BA$ is the anti-commutator.

(a) *Show that*

- $[AB, C] = A[B, C] + [A, C]B$,
- $[AB, C] = A\{B, C\} - \{A, C\}B$.

(2 points)

(b) Let U be a unitary operator. *Show that* $U[A, B]U^\dagger = [UAU^\dagger, UBU^\dagger]$. **(2 points)**

(c) Suppose that $[A, B] = z\hat{1}$, where z is a complex number. *Show that*

$$[A, B^n] = znB^{n-1}, \quad n = 1, 2, \ldots.$$

Hint: Use induction and exercise 1.2(a). **(2 points)**

(d) Let $[A, B] = z\hat{1}$ and let f be a function with a well defined Taylor expansion. *Show that*

$$[A, f(B)] = zf'(B),$$

where f' denotes the derivative of f. **(2 points)**

Remark: This result implies that if $[A, B] = 0$, then $[A, f(B)] = 0$.

(e) Let U be a unitary operator and f be a function with a well defined Taylor expansion. *Show that*

$$Uf(A)U^\dagger = f(UAU^\dagger)$$

for all operators A. **(1 point)**

[1] and preferably beyond.

[2] Tutors have been known to claim that they will commit suicide in class if they see too many falsely commuted operators, so please do not subject them to such cruelties.

1.3 Schrödinger picture and Heisenberg picture

The state of a quantum system evolves according to Schrödinger's equation $i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle$, for a time-independent Hamiltonian H. The solution can be written as $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ for the initial state $|\psi(0)\rangle$, and the unitary evolution operator $U(t) = e^{-itH/\hbar}$. It follows that the expectation value of an observable A evolves like $\langle A\rangle(t) = \langle\psi(t)|A|\psi(t)\rangle$. This way of describing the evolution is often referred to as the "Schrödinger picture", where the observables are time-independent, while the states evolve. In the "Heisenberg picture" it is instead the observables that evolve, as $A^H(t) = U(t)^\dagger A U(t)$, and the states that are time-independent. The equivalence of the Schrödinger and Heisenberg picture follows by the relation

$$\langle A\rangle(t) = \langle\psi(t)|A|\psi(t)\rangle = \langle\psi(0)|U(t)^\dagger A U(t)|\psi(0)\rangle = \langle\psi(0)|A^H(t)|\psi(0)\rangle.$$

(a) *Show that the evolution operator $U(t)$ and the Hamiltonian H commute.* Next, use this to show that the Hamilton operator looks the same in the Schrödinger picture as does in the Heisenberg picture. In other words, *show that $H^H = H$.*

Hint: Think of exercise 1.2(d). **(2 points)**

(b) *If A is a time-independent observable (in the Schrödinger picture), show that in the Heisenberg picture it obeys Heisenberg's equation of motion*

$$i\hbar\frac{d}{dt}A^H = -[H, A^H].$$

(4 points)

(c) A quantum particle with mass m moves in a one-dimensional potential $V(x)$, which has a well defined Taylor series. This is described by the Hamilton operator $H = \frac{1}{2m}P^2 + V(X)$, where X and P are the position and momentum operators. *Derive the equations of motion of the position and momentum operator in the Heisenberg picture, i.e., derive the equations for X^H and P^H.*

Hint: The position and momentum operator satisfy the canonical commutation relation $[X, P] = i\hbar\hat{1}$. Think of the various properties that we proved in exercise 1.2. Could some of those be used? **(4 points)**

Comment: The purpose of this whole exercise sheet is to remind about various properties and concepts that will be useful for us.