Advanced Quantum Mechanics

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Exercise sheet 10 (Due: Monday December, 16$^{\text{th}}$.)

10.1 The Klein-Gordon equation for free particles

In the lecture we introduced the KG equation for free particles

$$\frac{1}{c^2}\partial_t^2\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0. \tag{1}$$

We first treated ψ as a wave-function, and later as an operator acting on the Hilbert space of a quantum field. In this exercise we will follow the wave-equation scenario, while in 10.2 we will work with the field version.

Here we wish to analyze the behavior of the KG equation in the non-relativistic limit. To this end, it is useful to rewrite the KG equation, which is second order in time, into two coupled equations that are first order in time[1].

(a) For any function[2] $\psi(t,\vec{r})$ we define two new functions $\phi(t,\vec{r})$ and $\chi(t,\vec{r})$ by

$$\begin{aligned}\phi(t,\vec{r}) =& \frac{1}{2}\psi(t,\vec{r}) + \frac{i\hbar}{2mc^2}\partial_t\psi(t,\vec{r}),\\ \chi(t,\vec{r}) =& \frac{1}{2}\psi(t,\vec{r}) - \frac{i\hbar}{2mc^2}\partial_t\psi(t,\vec{r}).\end{aligned} \tag{2}$$

Show that if ψ satisfies the KG equation, then ϕ and χ satisfy

$$i\hbar\partial_t\phi = -\frac{\hbar^2}{2m}\nabla^2(\phi+\chi) + mc^2\phi, \qquad i\hbar\partial_t\chi = \frac{\hbar^2}{2m}\nabla^2(\phi+\chi) - mc^2\chi. \tag{3}$$

Hint: At some point it can be useful to invert (2) and express ψ and $\partial_t\psi$ in terms of ϕ and χ. **(3 points)**

Remark: To show that the coupled equations (3) are equivalent to the KG equation (1), we should strictly speaking also show that if ϕ and χ satisfy (3), then $\psi = \phi + \chi$ satisfy (1). However, we skip this (although nothing would prevent you from showing it anyway :-)

(b) *Make an ansatz of the form*

$$\begin{bmatrix}\phi(t,\vec{r})\\ \chi(t,\vec{r})\end{bmatrix} = e^{-\frac{i}{\hbar}(Et-\vec{p}\cdot\vec{r})}\begin{bmatrix}a\\ b\end{bmatrix}, \quad E\in\mathbb{R}, \quad a,b\in\mathbb{C},$$

in (3) and show that this leads to an eigenvalue problem of the form $M\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right] = E\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right]$ for a 2×2 matrix M. Determine M, and find its eigenvalues, and argue why we should expect to get these eigenvalues. **(4 points)**

(c) *Determine the eigenvectors of M, and combine this with (b) to write down the corresponding solutions to (3) as*

$$\Psi_\pm(t,\vec{r}) = \begin{bmatrix}\phi(t,\vec{r})\\ \chi(t,\vec{r})\end{bmatrix}_\pm = \mathcal{N}e^{-\frac{i}{\hbar}(\pm E_{\vec{p}}t-\vec{p}\cdot\vec{r})}\begin{bmatrix}mc^2 \pm E_{\vec{p}}\\ mc^2 - \pm E_{\vec{p}}\end{bmatrix},$$

[1] Transformations between equations with higher order derivatives, and coupled equations with lower order derivatives is a common trick that can be rather useful.

[2] Well, for any sufficiently smooth function.

where $E_{\vec{p}} = \sqrt{c^2p^2 + m_0^2c^4}$, $\quad p = \|\vec{p}\|$, *with* m_0 *being the rest mass of the particle.* The quantity $\mathcal{N}$ is a normalization factor that we do not bother to determine.

(2 points)

(d) We can conclude from (b) and (c) that the free Klein-Gordon equation has two types of plane-wave solutions. One class where the energy is positive, and one where the energy is negative. Often these are somewhat vaguely associated to particles and anti-particles. (With a field theoretic treatment, as in 10.2, we do not have to be vague any more.) A rather relevant question is how these solutions behave in the low energy limit, i.e., when speeds are not relativistic. In particular one can note that the two components of the vector $\begin{bmatrix} mc^2 \pm E_{\vec{p}} \\ mc^2 - \pm E_{\vec{p}} \end{bmatrix}$ determines the relative weight between ϕ and χ in the solutions $\Psi_{\pm}$.

- *What are the weights of the two components* ϕ *and* χ *for the positive and negative plane-waves* Ψ_+ *and* Ψ_- *in the case when the momentum is zero?*
- *Expand* $E_{\vec{p}}$ *up to the first order in* $\frac{p^2}{m^2c^2}$. *You will get two energy terms. Interpret these two terms.*
- *What happens to* $\begin{bmatrix} mc^2 \pm E_{\vec{p}} \\ mc^2 - \pm E_{\vec{p}} \end{bmatrix}$ *for small* $\frac{p^2}{m^2c^2}$*? What does that mean for the relative weight of* ϕ *and* χ *in the solutions* $\Psi_{\pm}$*?*

(4 points)

(e) *Argue that the evolution of positive energy states in the non-relativistic regime (i.e. for small* $\frac{p^2}{m^2c^2}$*) is approximately governed by a Schrödinger equation.* In other words, show that we in the non-relativistic limit regain what we are used to in non-relativistic quantum mechanics.

Hint: Consider the results in (d) for this regime. Which terms in (3) are going to be large, and which are going to be small? Be bold and only consider the equation for the dominant term, and put the small things to zero in that equation. Note that this problem to its very nature is rather hand-wavy, so we do not expect any particularly rigorous arguments. **(3 points)**

10.2 Two conserved quantities of the KG equation that actually are the same

Here we turn to the field-version of the Klein-Gordon equation, where the wave-functions are re-interpreted as the field operators

$$\hat{\Psi}(t, \vec{x}) = \frac{1}{\sqrt{L^3}} \sum_{\vec{p}} \sqrt{\frac{1}{2E_{\vec{p}}}} \Big(e^{-i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} a_{\vec{p}} + e^{i(E_{\vec{p}}t - \vec{p}\cdot\vec{x})} b_{\vec{p}}^\dagger \Big), \tag{4}$$

where we, like in the lecture, have put $\hbar = 1$ and $c = 1$. Moreover, $a_{\vec{p}}, a_{\vec{p}}^\dagger$ and $b_{\vec{p}}, b_{\vec{p}}^\dagger$ are bosonic annihilation and creations operators for particles and anti-particles. In the lecture we obtained a conserved quantity, which in the field version takes the form

$$Q(t) = i \int_V \Big(\hat{\Psi}^\dagger (\partial_t \hat{\Psi}) - (\partial_t \hat{\Psi}^\dagger) \hat{\Psi} \Big) \, d^3r. \tag{5}$$

Show that

$$Q(t) = \sum_{\vec{p}} a_{\vec{p}}^\dagger a_{\vec{p}} - \sum_{\vec{p}} b_{\vec{p}} b_{\vec{p}}^\dagger. \tag{6}$$

(4 points)

Remark: In the previous version of this exercise, I incorrectly asked you to show that $Q(t)$ is identical to $\hat{Q} = \sum_{\vec{p}} a_{\vec{p}}^\dagger a_{\vec{p}} - \sum_{\vec{p}} b_{\vec{p}}^\dagger b_{\vec{p}}$. Note that $b_{\vec{p}} b_{\vec{p}}^\dagger = b_{\vec{p}}^\dagger b_{\vec{p}} + \hat{1}$, so in each summand, the difference is an identity.