Advanced Quantum Mechanics

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Exercise sheet 11 (Due: Monday January, 13$^{\text{th}}$.)

11.1 β and α^i versus γ^μ

In the lecture we defined the matrices $\beta, \alpha^1, \alpha^2, \alpha^3$ in terms of the $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ as

$$\beta = (\gamma^0)^{-1}, \quad \alpha^i = (\gamma^0)^{-1}\gamma^i, \quad i = 1, 2, 3. \tag{1}$$

We also found that, in order for the Dirac equation to respect the relativistic energy-momentum relation, we necessarily must have

$$\begin{aligned} &\beta^2 = \mathbb{I}, \quad (\alpha^i)^2 = \mathbb{I}, \quad i = 1, 2, 3, \\ &\alpha^i\alpha^j = -\alpha^j\alpha^i, \quad i \neq j, \quad i, j = 1, 2, 3, \\ &\alpha^i\beta = -\beta\alpha^i, \quad i = 1, 2, 3, \end{aligned} \tag{2}$$

where $\mathbb{I}$ is the identity matrix. In the lecture it was claimed that these conditions can be equivalently expressed in terms of the γ-matrices as

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}, \quad \mu, \nu = 0, 1, 2, 3, \tag{3}$$

where $\{\cdot, \cdot\}$ is the anti-commutator, and where $g^{\mu\nu}$ is such that $g^{00} = 1$, $g^{11} = g^{22} = g^{33} = -1$, and is zero otherwise. *Show that (2) and (3) are equivalent.*

Hint: You need to show that if β, α^i satisfy (2) then γ^μ satisfy (3), and conversely that if γ^μ satisfy (3) then β, α^i satisfy (2). You can assume that these matrices are invertible.

(6 points)

11.2 Equation of continuity for the Dirac equation

In the lecture we showed that the solutions to the Klein-Gordon equation satisfies a continuity equation. Here we show that the Dirac equation also possesses a continuity equation. The Dirac equation can be written

$$i\frac{\partial}{\partial t}\Psi = -i\alpha^j\partial_j\Psi + m_0\beta\Psi,$$

where we apply the summation convention, and where we have put $\hbar = 1$ and $c = 1$. One should keep in mind that Ψ is a spinor (it is a column vector with four components). We define the density $j^0 = \Psi^\dagger\Psi$ and the current density $\vec{j} = \Psi^\dagger\vec{\alpha}\Psi$, which means $(j^1, j^2, j^3) = (\Psi^\dagger\alpha^1\Psi, \Psi^\dagger\alpha^2\Psi, \Psi^\dagger\alpha^3\Psi)$.

(a) *Show that* $\frac{\partial}{\partial t}j^0 + \nabla\cdot\vec{j} = 0$. **(4 points)**

(b) Recall that for the Klein-Gordon equation, the density (and charge) could become negative. *For* $j^0 = \Psi^\dagger\Psi$ *show that the expectation value is such that* $\mathrm{Tr}(j^0\hat{\rho}) \geq 0$ *for any density operator* $\hat{\rho}$. In other words, the expected density can never be negative.

(2 points)

Remark: It is a bit unfortunate that we both have a "density" j^0 and a "density operator" $\hat{\rho}$. Such clashes in terminology do unfortunately happen now and then. (Often it is due to independent development of terminology in different subfields.)

11.3 Angular momentum and the Dirac equation

The non-relativistic Hamiltonian of a free particle (i.e. $H = \vec{p}^2/(2m)$) commutes with the (orbital) angular momentum operator $\vec{L} = \vec{r} \times \vec{p}$ (like it would for any rotationally symmetric system). Here we shall see that this is not the case for the Dirac Hamiltonian, and that we are more or less forced to include spin in order to regain conservation of angular momentum. We put $c = 1$ and $\hbar = 1$.

(a) *Show that the orbital angular momentum operator $\vec{L} = \vec{r} \times \vec{p}$ does* not *commute with the Dirac Hamiltonian $H = \vec{\alpha} \cdot \vec{p} + \beta m_0$.*

Hint: Recall that $\vec{L}$ can be written component-wise as $L_l = \sum_{j,k} \epsilon_{ljk} r_j p_k$. To show that $[H, \vec{L}] \neq 0$ you have to show that at least one of the components of $\vec{L}$ does not commute with H. Note also that the matrices β and α^i commute with the position and momentum operators r_j and p_k (since they act on different spaces). The Levi-Civita symbol (or the completely anti-symmetric tensor) is defined such that $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$, $\epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1$, and is zero for all other values of the indices. **(4 points)**

(b) There are many different representations of the gamma matrices, and thus also of β and α^i. In the lecture we introduced the Dirac representation $\gamma^0 = \begin{bmatrix} \mathbb{I} & \\ & -\mathbb{I} \end{bmatrix}$, $\gamma^j = \begin{bmatrix} & \sigma_j \\ -\sigma_j & \end{bmatrix}$. By using (1) it follows that $\beta = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{bmatrix}$ and $\alpha^j = \begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix}$.

Let us now introduce the operator $\vec{S} = \frac{1}{2} \begin{bmatrix} \vec{\sigma} & \\ & \vec{\sigma} \end{bmatrix}$, with components

$$S_j = \frac{1}{2} \begin{bmatrix} \sigma_j & \\ & \sigma_j \end{bmatrix}, \quad j = 1, 2, 3,$$

with σ_1, σ_2, and σ_3 being the Pauli matrices. $\vec{S}$ can be interpreted as a spin-operator.

- *Show that the Dirac Hamiltonian H (in the Dirac representation) commutes with $\vec{L} + \vec{S}$.*
- *What is the physical interpretation of the operator $\vec{L} + \vec{S}$?*

Hint: Recall that the Pauli-operators satisfy the commutation relations $[\sigma_1, \sigma_2] = 2i\sigma_3$ and similar for all cyclic permutations. A more concise way to describe this is $[\sigma_j, \sigma_k] = 2i \sum_l \epsilon_{jkl} \sigma_l$.

(4 points)