Advanced Quantum Mechanics

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Exercise sheet 12 (Due: Monday January, 20$^{\text{th}}$.)

12.1 Solutions to the Dirac equation

In the lecture we sketched the derivation of the plane-wave solutions to the Dirac equation in the Weyl representation. In this exercise, we will work out the solutions in detail, but instead in the Dirac representation, and we will classify the solutions in terms of energy and helicity. In this exercise we set $\hbar = 1$ and $c = 1$.

We wish to find solutions to the Dirac equation

$$i\partial_t \Psi_{\vec{p}} = H\Psi_{\vec{p}}, \quad H = -i\sum_{k=1}^{3} \alpha^k \partial_k + \beta m, \tag{1}$$

and we use the representation where

$$\alpha^k = \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{bmatrix}. \tag{2}$$

(a) We make an ansatz in the form of plane waves

$$\Psi_{\vec{p}}(t,\vec{r}) = we^{-i[E_{\vec{p}}t - \vec{p}\cdot\vec{r}]}, \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}. \tag{3}$$

By inserting $\Psi_{\vec{p}}$ into (1), the latter reduces to an eigenvalue problem for the vectors w, i.e, $Mw = E_{\vec{p}}w$. It turns out that M conveniently can be written as block-matrix on the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{4}$$

where A, B, C, D are 2×2 matrices. *Determine the matrices A, B, C, D.* (It is enough to express these matrices in terms of the Pauli matrices σ_k and the identity matrix $\mathbb{I}$.)

(3 points)

(b) Let us now consider the special case $\vec{p} = p_z\hat{e}_z$, with $p_z > 0$, i.e., the case where we have plane waves in the positive z-direction (and $\hat{e}_z$ denotes the unit vector in the positive z-direction). *In this special case, show that*

$$M = \begin{bmatrix} m & 0 & p_z & 0 \\ 0 & m & 0 & -p_z \\ p_z & 0 & -m & 0 \\ 0 & -p_z & 0 & -m \end{bmatrix}. \tag{5}$$

Hint: You might be able to solve this problem even if you do not manage (a).

(2 points)

(c) *What are the eigenvalues E_{p_z} of (5)? What are the degeneracies of these eigenvalues?*
Hint: by looking at (5), one can see that it block-diagonalizes.

(3 points)

(d) Consider the vectors

$$w_a = \begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{K+m} \\ 0 \end{bmatrix}, \quad w_b = \begin{bmatrix} -\frac{p_z}{K+m} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad w_c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\frac{p_z}{K+m} \end{bmatrix}, \quad w_d = \begin{bmatrix} 0 \\ \frac{p_z}{K+m} \\ 0 \\ 1 \end{bmatrix}, \tag{6}$$

$$K = \sqrt{m^2 + p_z^2}.$$

Show that these are (not necessarily normalized) eigenvectors to (5) and determine the corresponding energy eigenvalues E_{p_z}. In particular, note which of the eigenvectors that correspond to positive and negative energies. **(4 points)**

(e) We can define a helicity operator as[1]

$$\tilde{h} = \sum_{k=1}^{3} \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix} p_k = -i \sum_{k=1}^{3} \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix} \partial_k \tag{7}$$

Show that $\Psi_{p_z,\xi} = w_\xi e^{-i[E_{p_z,\xi}t - \vec{p}\cdot\vec{r}]}$ for $\xi = a, b, c, d$, are eigenfunctions to $\tilde{h}$, and determine the corresponding eigenvalues. In particular, note which of the eigenvectors that correspond to positive and negative helicities. (Recall that we assume $p_z > 0$.)

(4 points)

Remark: By this we have classified the plane-wave solutions $\Psi_{p_z,\xi} = w_\xi e^{-i[E_{p_z,\xi}t - p_z z]}$ for $\xi = a, b, c, d$, such that each of them has a unique label in terms of the momentum p_z, the sign of the energy, and the sign of the helicity.

(f) In exercise 11.2 we introduced the (maybe somewhat mysterious looking) current density $\vec{j} = \Psi^\dagger \vec{\alpha} \Psi$, meaning $(j^1, j^2, j^3) = (\Psi^\dagger \alpha^1 \Psi, \Psi^\dagger \alpha^2 \Psi, \Psi^\dagger \alpha^3 \Psi)$. Here we shall take a look at the current densities of our plane wave solutions.

Calculate the current densities of $\Psi_{p_z,\xi} = w_\xi e^{-i[E_{p_z,\xi}t - p_z z]}$ for $\xi = a, b, c, d$. How does the direction of the current density compare to the momentum $\vec{p} = p_z \hat{e}_z$? Does it depend on the energy being positive or negative? (Recall that we assume $p_z > 0$.) **(4 points)**

Recall that the Pauli matrices are

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{8}$$

[1] In the lecture we considered $h = \tilde{h}/\|\vec{p}\|$, but in the current case it is slightly less messy to instead use $\tilde{h}$.