Advanced Quantum Mechanics

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Exercise sheet 13 (Due: Monday January, 27$^{\text{th}}$.)

This will be counted as a bonus sheet. So in total you will need 120 points (including the points that you get on this sheet) in order to do the exam.

13.1 Scattering from an oscillatory potential

For a potential V that is spherically symmetric, the first-order Born approximation of the scattering amplitude becomes

$$f_B^{(1)}(q) = -\frac{2m}{\hbar^2 q}\int_0^{+\infty} rV(r)\sin(qr)dr,$$

where $q = \|\vec{k}_{\text{out}} - \vec{k}_{\text{in}}\|$, with $\vec{k}_{\text{in}}$ being the wave-vector of the incoming wave, and $\vec{k}_{\text{out}}$ the wave-vector of the scattered wave.

Suppose that the radial dependence of the potential is $V(r) = \sin(ar)e^{-br}$, for $a > 0$ and $b > 0$. In other words, the potential oscillates, but the oscillations go to zero exponentially fast as r approaches infinity.

(a) *Show that the scattering amplitude in the first order Born approximation is*

$$f_B^{(1)}(q) = \frac{m}{\hbar^2 q}\Big[\frac{b^2-(q+a)^2}{[b^2+(q+a)^2]^2} - \frac{b^2-(q-a)^2}{[b^2+(q-a)^2]^2}\Big]. \tag{1}$$

Hint: Keep in mind the relation $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$. It can be convenient to use

$$\int_0^{+\infty} xe^{\gamma x}dx = \frac{1}{\gamma^2}, \quad \text{for} \quad \text{Re}(\gamma) < 0.$$

(4 points)

(b) If we let $b \to 0$ in (1), i.e., if we take the limit of an infinitely slow decay of the potential with the distance, one finds that the scattering amplitude goes to infinity for a certain value of q (apart from $q = 0$). *What value is that?* **(2 points)**

(c) *What do you think is the reason for there being such a strong scattering at the value you found in (b)?* **(2 points)**

13.2 Scattering from a row of delta potentials

The general expression for the first order Born-approximation of the scattering amplitude is

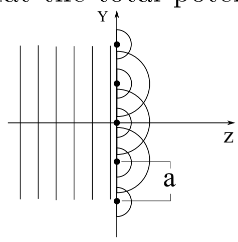
$$f_B^{(1)}(\vec{q}) = -\frac{m}{2\pi\hbar^2}\int d^3r V(\vec{r})e^{-i\vec{q}\cdot\vec{r}}. \tag{2}$$

The general relation between the differential scattering cross section $\frac{d\sigma^{(1)}}{d\Omega}$ and the scattering amplitude $f_B^{(1)}(\vec{q})$ is given by

$$\frac{d\sigma^{(1)}}{d\Omega} = |f_B^{(1)}(\vec{q})|^2. \tag{3}$$

Here we consider a model for the elastic scattering of particles (e.g. of neutrons) by the nuclei of a linear molecule, or by a row of trapped ions or atoms. Suppose that an incoming particle of mass m, with wave-number k in the direction $\vec{e}_z$, is scattered by a row of delta-potentials, distributed along the y-axis in such a way that the total potential is

$$V(x,y,z) = \gamma\sum_{n=0}^{N-1}\delta(z)\delta(x)\delta\big(y-an\big),$$



where $a > 0$ and $\gamma > 0$.

(a) *Show that the scattering amplitude in the first-order Born approximation is*

$$f_B^{(1)}(\vec{q}) = -\frac{m\gamma}{2\pi\hbar^2}e^{-iaq_y(N-1)/2}\frac{\sin(aq_yN/2)}{\sin(aq_y/2)},$$

where q_y is the y-component of $\vec{q} = (q_x, q_y, q_z) = \vec{k}_{out} - \vec{k}_{in}$. **(4 points)**

(b) Let us assume that $\vec{k}_{\text{in}} = k\vec{e}_z$, $k > 0$, and that we use spherical coordinates $(x, y, z) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$. *Show that the differential scattering cross section in direction (θ,φ) is*

$$\frac{d\sigma^{(1)}}{d\Omega} = \frac{m^2\gamma^2}{4\pi^2\hbar^4}\frac{\sin^2(\frac{akN}{2}\sin\theta\sin\varphi)}{\sin^2(\frac{ak}{2}\sin\theta\sin\varphi)}. \tag{4}$$

Hint: Since the scattering is elastic, we have $\|\vec{k}_{\text{out}}\| = \|\vec{k}_{\text{in}}\|$.

(4 points)

(c) For the result in (b)

- *If you let $N = 1$, how does the differential scattering cross section depend on the angles θ and φ?*
- Suppose that N and a are fixed. Let us now consider the limit of very small k, i.e., very long wavelengths. *Show that*

$$\frac{d\sigma^{(1)}}{d\Omega} = \alpha + O(k^2) \tag{5}$$

What is the value of α, and how does it depend on N, θ and φ?

(4 points)