Advanced Quantum Mechanics

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Exercise sheet 2 (Due: Monday October, 21st.)

2.1 Calculating with traces

Recall that when we want to evaluate the trace, we can do it as $\mathrm{Tr}(Q) = \sum_n \langle n|Q|n\rangle$, where $\{|n\rangle\}_n$ is *any* orthonormal basis of $\mathcal{H}$.[1] In this exercise we establish two convenient properties concerning calculations with traces.

(a) Let $|\psi\rangle, |\chi\rangle \in \mathcal{H}$. *Show that*

$$\mathrm{Tr}(|\psi\rangle\langle\chi|) = \langle\chi|\psi\rangle.$$

(1 point)

(b) Let $Q : \mathcal{H} \to \mathcal{H}$ be linear, and let $|\psi\rangle \in \mathcal{H}$. *Show that*

$$\langle\psi|Q|\psi\rangle = \mathrm{Tr}(Q|\psi\rangle\langle\psi|).$$

(1 point)

Remark: We used this in the lecture without proving it.

2.2 Two distinguishable systems

Consider two distinguishable quantum systems with Hilbert spaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$, respectively. As we know from the lecture, the Hilbert space of the joint system is $\mathcal{H}^{(1,2)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$.

Suppose that the two systems are in a pure state, where the joint state vector is

$$|\psi\rangle = \frac{1}{\sqrt{3}}|\alpha_1\rangle \otimes |\beta_1\rangle + \sqrt{\frac{2}{3}}|\alpha_2\rangle \otimes |\beta_2\rangle, \tag{1}$$

where $|\alpha_1\rangle, |\alpha_2\rangle$ is an orthonormal set in $\mathcal{H}^{(1)}$, and $|\beta_1\rangle, |\beta_2\rangle$ is an orthonormal set in $\mathcal{H}^{(2)}$.

(a) *What is the reduced density operator of particle* 1*?*
Determine whether the reduced state is mixed or pure. **(2 points)**

(b) On system 1, we have the following observable

$$\begin{aligned} Q_1 = &5\big(|\alpha_1\rangle + |\alpha_2\rangle\big)\big(\langle\alpha_1| + \langle\alpha_2|\big) \\ &- \frac{1}{2}\big(|\alpha_1\rangle - |\alpha_2\rangle\big)\big(\langle\alpha_1| - \langle\alpha_2|\big). \end{aligned} \tag{2}$$

Why can we say that Q_1 is an observable?
What is the expectation value of Q_1? **(3 points)**

(c) *What is the probability that we detect the largest eigenvalue of Q_1, if we measure Q_1?* **(3 points)**

[1] If the Hilbert space is infinite-dimensional, then one needs to put restrictions on Q for $\mathrm{Tr}(Q)$ to make sense, but we will not bother about this here. However, if you are curious, you can search for "trace class operators".

2.3 Symmetric and anti-symmetric subspaces for two distinguishable particles

Suppose that we have two *distinguishable* particles, each with Hilbert space $\mathcal{H}$. The total Hilbert space of the two particles is $\mathcal{H} \otimes \mathcal{H}$. If $\{|n\rangle\}_{n=1}^{N}$ is an orthonormal basis of $\mathcal{H}$, then $\{|n\rangle_1 \otimes |n'\rangle_2\}_{n,n'}$ is an orthonormal basis of $\mathcal{H} \otimes \mathcal{H}$. We are interested in what happens when we swap the states of the two particles, and we can we can do this via the permutation operator[2]

$$P_{12} = \sum_{n=1}^{N} \sum_{n'=1}^{N} |n\rangle\langle n'| \otimes |n'\rangle\langle n|.$$

We say that a state $|\xi\rangle \in \mathcal{H}\otimes\mathcal{H}$ is symmetric if $P_{12}|\xi\rangle = |\xi\rangle$, and that it is anti-symmetric if $P_{12}|\xi\rangle = -|\xi\rangle$.

(a) *Show that P_{12} is Hermitian, unitary, and satisfies $P_{12}^2 = \hat{1}_{\mathcal{H}} \otimes \hat{1}_{\mathcal{H}}$.* **(2 points)**

(b) *Show that $P_{12}|\psi\rangle \otimes |\chi\rangle = |\chi\rangle \otimes |\psi\rangle$.* **(1 point)**

Remark: This suggests that P_{12} indeed swaps the states of the two particles.

(c) Define the two operators $P_S = \frac{1}{2}(\hat{1} + P_{12})$ and $P_A = \frac{1}{2}(\hat{1} - P_{12})$.
Show that P_S and P_A are projectors.[3] **(1 point)**

Remark: Recall that to every projector corresponds a subspace, and vice versa. Hence P_S corresponds to a subspace $\mathcal{L}_S \subset \mathcal{H}\otimes\mathcal{H}$, and P_A corresponds to a subspace $\mathcal{L}_A \subset \mathcal{H}\otimes\mathcal{H}$.

(d) *Show that the projectors P_S and P_A are orthogonal to each other, in the sense that $P_S P_A = P_A P_S = 0$.*
Show also that P_S and P_A are complementary, in the sense that $P_S + P_A = \hat{1}$.
(2 points)

Remark: We conclude from this that every element of $\mathcal{H} \otimes \mathcal{H}$ can be decomposed as an orthogonal sum of one element in $\mathcal{L}_S$ and one element in $\mathcal{L}_A$. In other words, $\mathcal{L}_S$ is orthogonal to $\mathcal{L}_A$, and $\mathcal{L}_S \oplus \mathcal{L}_A = \mathcal{H} \otimes \mathcal{H}$.

(e) *Show that $P_{12}P_S = P_S$ and that $P_{12}P_A = -P_A$.* **(1 point)**

Remark: This means that all elements in $\mathcal{L}_S$ are symmetric, while all elements of $\mathcal{L}_A$ are anti-symmetric. This also means that if $|\xi\rangle$ is any element in $\mathcal{H} \otimes \mathcal{H}$, then we know that $P_S|\xi\rangle$ is symmetric, and $P_A|\xi\rangle$ is anti-symmetric. Hence, this is a method to construct symmetric or anti-symmetric states.

(f) *What are the eigenvalues of P_{12}, and what are the dimensions of the corresponding eigenspaces?*

Hint: There is no need for any long calculation. You already have all the components that you need. First show that $P_{12} = P_S - P_A$. Also, recall that if a projector P projects onto a subspace $\mathcal{L}$, then $\dim \mathcal{L} = \mathrm{Tr}(P)$. **(3 points)**

Remark on the whole exercise: We have discovered that the space of two distinguishable particles (with isomorphic Hilbert spaces) decomposes into the symmetric subspace $\mathcal{L}_S$ and the anti-symmetric subspace $\mathcal{L}_A$. Note that $\mathcal{L}_S$ would be the state space if the two particles were identical bosons, and $\mathcal{L}_A$ would be the state space is they would be identical fermions. Hence, the Hilbert space of the two distinguishable particles decomposes into that of two bosons and that of two fermions. This nice picture does unfortunately not survive if we have more than two particles, since then more subspaces pop up, as we perhaps will see is some other exercise.

[2] Also referred to as the exchange operator, or the swap operator.

[3] What we here refer to as a "projector", a mathematician typically would refer to as an "orthogonal projector".