Advanced Quantum Mechanics

D. Gross, J. Åberg

Exercise sheet 4 (Due: Monday November, 4$^{\text{th}}$.)

4.1 Labeling of particles versus the occupation number representation

So far we have treated identical particles by labeling them, and followed the rule that the state-vector has to be totally symmetric or totally anti-symmetric depending on whether we have bosons or fermions. In the Fock representation (or occupation number representation) we do instead specify how a collection of (orthogonal) single-particle states are occupied. This method of describing the system does not involve any labeling of particles, and there is no need for symmetrization or anti-symmetrization. As you most likely will notice, this method tends to give a more compact notation. In the following, let $|\phi_\alpha\rangle$, $|\phi_\beta\rangle$, $|\phi_\gamma\rangle$, $|\phi_\delta\rangle$, and $|\phi_\epsilon\rangle$ be a collection of (orthonormal) single particle state-vectors.

(a) Let $|0{,}2{,}1{,}0{,}1\rangle$ be a bosonic Fock state-vector with respect to the ordering of the single-particle states indicated above. (Hence, there are two quanta in state ϕ_β, one in ϕ_γ and one in ϕ_ϵ.) *Determine the multi-particle wave-function* $\psi(x_1,x_2,x_3,x_4) = \langle x_1,x_2,x_3,x_4|0{,}2{,}1{,}0{,}1\rangle$ *in terms of the wave functions* $\phi_\alpha(x) = \langle x|\phi_\alpha\rangle$, $\phi_\beta(x) = \langle x|\phi_\beta\rangle$, *etc.* **(2 points)**

(b) Consider the following totally anti-symmetric state-vector over three particles

$$|\psi_a\rangle = \frac{1}{\sqrt{6}}\Big(|\phi_\alpha\rangle_1|\phi_\beta\rangle_2|\phi_\delta\rangle_3 + |\phi_\delta\rangle_1|\phi_\alpha\rangle_2|\phi_\beta\rangle_3 + |\phi_\beta\rangle_1|\phi_\delta\rangle_2|\phi_\alpha\rangle_3 \\ - |\phi_\beta\rangle_1|\phi_\alpha\rangle_2|\phi_\delta\rangle_3 - |\phi_\delta\rangle_1|\phi_\beta\rangle_2|\phi_\alpha\rangle_3 - |\phi_\alpha\rangle_1|\phi_\delta\rangle_2|\phi_\beta\rangle_3\Big).$$

Write down the corresponding fermionic Fock-representation, with respect to the ordering of the single-particle states indicated above. **(2 points)**

4.2 Transformations of creation and annihilation operators

Suppose that we have a collection of annihilation operators $a_1,\ldots,a_K$ (corresponding to orthogonal single-particle states), and imagine that we create a new collection of operators $b_1,\ldots,b_K$ via $b_l = \sum_{k=1}^{K} Q_{l,k}a_k$ for $l = 1,\ldots,K$, where Q is a complex K times K matrix $Q \in \mathbb{C}^{K\times K}$.

(a) Assume that $a_1,\ldots,a_K$ satisfy the bosonic commutation relations. *Find the necessary and sufficient conditions on Q for $b_1,\ldots,b_K$ to satisfy the bosonic commutation relations.* **(3 points)**

Remark: A change of (orthonormal) basis among the single-particle states (to which the annihilation operators are associated) causes a transformation of this type. It may also correspond to an active physical evolution, as we shall see in exercise 4.3.

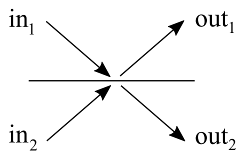
(b) In the previous exercise we investigated transformations that only mix the collection of annihilation operators within themselves (and in parallel mix the creation operators). However, one can consider an even more general type of transformation that combines both annihilation and creation operators. For the sake of simplicity we here only consider the transformation of a single pair $a, a^\dagger$ into a new pair $b, b^\dagger$. Let $a, a^\dagger$ be a bosonic annihilation and creation operator, and let $A, B \in \mathbb{C}$. *Find the necessary and sufficient conditions on A and B such that $b = Aa + Ba^\dagger$ and $b^\dagger = A^*a^\dagger + B^*a$, are bosonic annihilation and creation operators.* **(3 points)**

Remark: This type of mappings is often called Bogoliubov transformations, and can be used to diagonalise certain types of many-body Hamiltonians.

4.3 Beam-splitters (or the social life of fermions and bosons)

A beam-splitter is an optical device that acts like a half-transparent mirror that partially reflects and partially lets photons through. We can model this via a unitary operator U that maps the state of two input modes in_1, in_2 to the state of two output modes out_1, out_2. When a photon impinges on the beam-splitter, e.g. from in_1, one obtains a superposition of the photon propagating in out_1 and out_2, and similarly if the photon instead comes from in_2. This can be written in the occupation number representation as

$$\begin{aligned} U|1_{\mathrm{in}_1}, 0_{\mathrm{in}_2}\rangle &= \frac{1}{\sqrt{2}}(|1_{\mathrm{out}_1}, 0_{\mathrm{out}_2}\rangle + |0_{\mathrm{out}_1}, 1_{\mathrm{out}_2}\rangle), \\ U|0_{\mathrm{in}_1}, 1_{\mathrm{in}_2}\rangle &= \frac{1}{\sqrt{2}}(|1_{\mathrm{out}_1}, 0_{\mathrm{out}_2}\rangle - |0_{\mathrm{out}_1}, 1_{\mathrm{out}_2}\rangle). \end{aligned} \tag{1}$$



(The sign difference between the two lines in (1) is necessary for U to be unitary.)

Equation (1) only tells us what happens if we send a single particle onto the beam-splitter, but what happens if we send several? In the following we shall investigate a more general model, where we instead specify how U acts on the creation operators $a^\dagger_{\mathrm{in}_1}$, $a^\dagger_{\mathrm{in}_2}$, $a^\dagger_{\mathrm{out}_1}$, $a^\dagger_{\mathrm{out}_2}$ of the input and output modes. We assume that

$$Ua^\dagger_{\mathrm{in}_1}U^\dagger = \frac{1}{\sqrt{2}}(a^\dagger_{\mathrm{out}_1} + a^\dagger_{\mathrm{out}_2}), \quad Ua^\dagger_{\mathrm{in}_2}U^\dagger = \frac{1}{\sqrt{2}}(a^\dagger_{\mathrm{out}_1} - a^\dagger_{\mathrm{out}_2}), \quad U|0_{\mathrm{in}_1}, 0_{\mathrm{in}_2}\rangle = |0_{\mathrm{out}_1}, 0_{\mathrm{out}_2}\rangle, \tag{2}$$

where the last equality says that if there are no particles in the input, then there will be no particles in the output. In an actual optical setting we deal with photons, and the creation operators would be bosonic. However, in the following we shall also consider the fermionic case, and a purely classical model.

(a) The first equality in (2) says that the creation of a particle in the input in_1 has the same effect as applying $\frac{1}{\sqrt{2}}(a^\dagger_{\mathrm{out}_1} + a^\dagger_{\mathrm{out}_2})$ to the output. This looks suspiciously like the the creation of the superposition in the first row of (1). *Confirm this suspicion by showing that (2) implies (1). Does it matter whether the particle is a boson or a fermion?*

Hint: Since U is unitary, it is the case that $U^\dagger U = \hat{1}$. **(2 points)**

(b) Let us now imagine a classical probabilistic model of the beam splitter, where a particle that impinges from in_1 with 50% probability ends up in out_1 and 50% probability ends up in out_2. The same happens if a particle enters from in_2. We moreover assume that particles do not interact with each other. Assume that one particle is sent through in_1 *and* another through in_2. *What is the probability that both particles end up in out_1? What is the probability that both end up in out_2? What is the probability that one ends up in out_1 and one in out_2?* **(2 points)**

(c) Suppose that we have two bosons in the input state $|1_{\mathrm{in}_1}, 1_{\mathrm{in}_2}\rangle$. *Use (2) to determine the output state $U|1_{in_1}, 1_{in_2}\rangle$. What is the probability that we would detect both bosons in out_1? What is the probability that both would be detected in out_2? What is the probability that one is detected in out_1 and one in out_2?* **(3 points)**

(d) Suppose that we instead have two *fermions* in the input state $|1_{\mathrm{in}_1}, 1_{\mathrm{in}_2}\rangle$. *Determine the output state $U|1_{in_1}, 1_{in_2}\rangle$. What is the probability that we would detect both fermions in out_1? What is the probability that both would be detected in out_2? What is the probability that one is detected in out_1 and one in out_2? Compare with the results in (b) and (c).* **(3 points)**

Remark: This exercise illustrates the phenomenon of "bunching", which means that bosons seem to like to cluster, as well as "anti-bunching", which means that fermions tend to avoid each other.