Advanced Quantum Mechanics

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Exercise sheet 5 (Due: Monday November, 11th.)

5.1 Conservation of particle-number for one- and two-body Hamiltonians

We know from the lecture that Hamiltonians for identical particles that contain single-particle terms, as well as two-particle terms, can be written on the form

$$H = \sum_{jk} T_{jk} a_j^\dagger a_k + \sum_{jkmn} F_{jkmn} a_j^\dagger a_k^\dagger a_n a_m.$$

In this exercise we will show that Hamiltonians of this form always commute with the total number operator, in both the bosonic and fermionic case. This means that the particle number is conserved, i.e., the dynamics induced by H does not change the total number of particles in the system.

(a) If $a_j, a_j^\dagger$ are bosonic annihilation and creation operators, *show that*

$$\begin{aligned}[a_j^\dagger a_k, a_l^\dagger a_l] =& a_j^\dagger a_l \delta_{k,l} - a_l^\dagger a_k \delta_{l,j},\\ [a_j^\dagger a_k^\dagger a_n a_m, a_l^\dagger a_l] =& a_j^\dagger a_k^\dagger a_n a_l \delta_{m,l} + a_j^\dagger a_k^\dagger a_m a_l \delta_{n,l} - a_l^\dagger a_j^\dagger a_n a_m \delta_{l,k} - a_l^\dagger a_k^\dagger a_n a_m \delta_{l,j}\end{aligned}$$

Hint: Use the relation $[AB, C] = A[B, C] + [A, C]B$ from exercise 1.1(a) as well as the analogous relation $[A, BC] = B[A, C] + [A, B]C$, and then apply the bosonic commutation relations. **(3 points)**

(b) *Show that the total number operator $N = \sum_l a_l^\dagger a_l$ commutes with H in the bosonic case.* **(2 points)**

(c) If $a_j, a_j^\dagger$ are fermionic annihilation and creation operators, *show that*

$$\begin{aligned}[a_j^\dagger a_k, a_l^\dagger a_l] =& a_j^\dagger a_l \delta_{k,l} - a_l^\dagger a_k \delta_{l,j},\\ [a_j^\dagger a_k^\dagger a_n a_m, a_l^\dagger a_l] =& a_j^\dagger a_k^\dagger a_n a_l \delta_{m,l} - a_j^\dagger a_k^\dagger a_m a_l \delta_{n,l} + a_l^\dagger a_j^\dagger a_n a_m \delta_{l,k} - a_l^\dagger a_k^\dagger a_n a_m \delta_{l,j}.\end{aligned}$$

Hint: Use $[AB, C] = A[B, C] + [A, C]B$, $[A, BC] = B[A, C] + [A, B]C$, and $[AB, C] = A\{B, C\} - \{A, C\}B$ (see exercise 1.1(a)), as well as the fermionic anti-commutation relations.

(3 points)

(d) *Show that the total number operator $N = \sum_l a_l^\dagger a_l$ commutes with H also in the fermionic case.* **(2 points)**

5.2 Finding the spectrum of a Hamiltonian by transformation of annihilation and creation operators

Consider a single bosonic mode, with annihilation and creation operators $a, a^\dagger$, on which we define the Hamiltonian

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}\hat{1}) + \frac{1}{2}\hbar(\Delta^* a^\dagger a^\dagger + \Delta a a),$$

where $\omega > |\Delta|$. We wish to find the spectrum of this Hamiltonian.

In exercise 4.2 we introduced the Bobogliubov transformation in the special case of a single pair of bosonic annihilation and creation operators $a, a^\dagger$. We found that the pair

$b, b^\dagger$, defined by $b = Aa + Ba^\dagger$, are also bosonic annihilation and creation operators if and only if $|A|^2 - |B|^2 = 1$. *Find such a transformation to bosonic* $b, b^\dagger$ *with real numbers* $\tilde{\omega}$ *and* q*, such that*

$$H = \hbar\tilde{\omega} b^\dagger b + \hbar q \hat{1}, \tag{1}$$

and use this to determine the spectrum of H. **(4 points)**

5.3 Jordan-Wigner transformation

Consider the following Hamiltonian (we put $\hbar = 1$) on a chain of spin-half particles

$$H = \frac{J_\perp}{2} \sum_k (S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+), \tag{2}$$

where $S_k^\pm = S_k^x \pm i S_k^y$, where (S_k^x, S_k^y, S_k^z) are the spin-half operators on site k. (The relation to the Pauli-spin operators are $S_k^x = \frac{1}{2}\sigma_k^x$, $S_k^y = \frac{1}{2}\sigma_k^y$, and $S_k^z = \frac{1}{2}\sigma_k^z$, although you will not need that here.) Here we shall use a trick, called the Jordan-Wigner transformation, to rewrite this as a Hamiltonian for non-interacting fermions with annihilation operators c_k and creation operators $c_k^\dagger$.

The spin operators on different sites commute, but fermionic operators anti-commute. To represent spins by fermion operators c_j and $c_j^\dagger$, the unwanted sign can be canceled via a Jordan-Wigner transformation. We define the operators

$$S_k^+ = c_k^\dagger e^{i\pi \sum_{j<k} \hat{n}_j}, \quad S_k^- = c_k e^{-i\pi \sum_{j<k} \hat{n}_j}, \quad S_k^z = c_k^\dagger c_k - \frac{1}{2}\hat{1}, \tag{3}$$

where $\hat{n}_j = c_j^\dagger c_j$ is the number operator at site j.

(a) *Show that the* S_k^+, S_k^- *and* S_k^z *defined in (3) satisfy* $[S_k^+, S_k^-] = 2S_k^z$.

Hint: Note that all $\hat{n}_j$ commute with each other, and also that $\hat{n}_j$ commutes with c_k whenever, $k \neq j$.

(2 points)

Remark: This shows that on each site we get the correct spin-operators.

(b) *Show that*

$$S_k^+ S_{k+1}^- = c_k^\dagger c_{k+1}, \quad S_k^- S_{k+1}^+ = c_{k+1}^\dagger c_k, \tag{4}$$

and use this to show that the spin-Hamiltonian (2) can be rewritten as the fermionic Hamiltonian

$$H = \frac{J_\perp}{2} \sum_k (c_k^\dagger c_{k+1} + c_{k+1}^\dagger c_k). \tag{5}$$

Hint: First prove that $c_k^\dagger \hat{n}_k = 0$ and $c_k \hat{n}_k = c_k$ (recall the fermionic anti-commutation relations). Next, use a Taylor expansion to determine what $c_k^\dagger e^{-i\pi\hat{n}_k}$ and $c_k e^{i\pi\hat{n}_k}$ are.

(2 points)

(c) Let the chain contain N sites. We consider a new collection of fermionic annihilation and creation operators s_l and $s_l^\dagger$ that are related by the discrete Fourier transform

$$c_k^\dagger = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} s_l^\dagger e^{-2\pi i l k/N}.$$

For the sake of simplicity we assume that we have periodic boundary conditions in (5). *Show that*

$$H = J_{\perp} \sum_{l=0}^{N-1} \cos\left(\frac{2\pi l}{N}\right) s_l^{\dagger} s_l, \tag{6}$$

which means that we have a collection of non-interacting fermions. **(2 points)**

Gold-star exercise

This is a gold-star exercise, which means that you get no points what so ever for it, but you get a gold-star!

In exercise (a) we showed that on each single site, the resulting operators behave as spin operators should. However, they should also commute for different sites, i.e., we should have $[S_k^+, S_{k'}^+] = 0$, $[S_k^+, S_{k'}^-] = 0$, $[S_k^-, S_{k'}^-] = 0$, whenever $k \neq k'$. It is maybe a bit much to show them all, but why not show that

$$[S_k^+, S_{k'}^-] = 0, \quad k \neq k'.$$