Advanced Quantum Mechanics

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Exercise sheet 6 (Due: Monday November, 18$^{\text{th}}$.)

6.1 Kitaev chain

Consider a chain of N sites (open boundary conditions). To each site l we associate a fermionic annihilation operator c_l and creation operator $c_l^\dagger$, where these satisfy the standard anti-commutation relations $\{c_l, c_{l'}\} = 0$, $\{c_l^\dagger, c_{l'}^\dagger\} = 0$ and $\{c_l, c_{l'}^\dagger\} = \delta_{l,l'}\hat{1}$. Consider the Hamiltonian

$$H = \sum_{l=1}^{N-1}(c_{l+1}^\dagger c_l + c_l^\dagger c_{l+1} + c_l^\dagger c_{l+1}^\dagger + c_{l+1}c_l). \tag{1}$$

Here we will make a sequence of transformations of this Hamiltonians, such that we get it on a simple form.

(a) Define the operators

$$\eta_l = c_l + c_l^\dagger, \quad \chi_l = \frac{1}{i}(c_l - c_l^\dagger). \tag{2}$$

Show that

$$\eta_l^\dagger = \eta_l, \quad \chi_l^\dagger = \chi_l, \tag{3}$$

and show η_l and χ_l satisfy the anti-commutation relations

$$\{\eta_l, \eta_{l'}\} = 2\delta_{l,l'}\hat{1}, \quad \{\chi_l, \chi_{l'}\} = 2\delta_{l,l'}\hat{1}, \quad \{\eta_l, \chi_{l'}\} = 0. \tag{4}$$

(3 points)

Remark: Particles that are characterized by these types of operators are often referred to as Majorana-fermions.

(b) *Rewrite the Hamiltonian (1) in terms of the operators η_l and χ_l, and simplify.*

Hint: By using the anti-commutation relation and gathering terms in (1) one can use (2) directly. Alternatively one can invert the relations in (2) to express c_l and $c_l^\dagger$ in terms of η_l and χ_l, and substitute in (1). Note that one can simplify quite a lot (which helps when solving d). **(2 points)**

(c) Now we create a new set of operators

$$d_l = \frac{1}{2}(\eta_{l+1} + i\chi_l), \quad d_l^\dagger = \frac{1}{2}(\eta_{l+1} - i\chi_l), \quad l = 1, \ldots, N-1. \tag{5}$$

Show that d_l and $d_l^\dagger$ satisfy the fermionic anti-commutation relations. **(3 points)**

(d) *Show that the Hamiltonian that you obtained in (a) can be put on the form*

$$H = \alpha\hat{1} + \beta\sum_{l=1}^{N-1} d_l^\dagger d_l, \tag{6}$$

and determine the coefficients α and β. **(2 points)**

(e) *What is the degeneracy of the ground state of H?* **(2 points)**

6.2 Gauge invariance

A common starting point for quantization of the electromagnetic field, is the classical electromagnetic field expressed in the Coulomb gauge. Here we recall some facts about the gauge symmetries of the classical electromagnetic field.

The electric and magnetic fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ depend on the scalar and vector potentials $\phi(\vec{x},t)$ and $\vec{A}(\vec{x},t)$ via the equations

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}. \tag{7}$$

(a) *Show that (the classical) fields $\vec{E}$ and $\vec{B}$ as in (7) stay invariant under the gauge transformation*

$$\phi' = \phi - \frac{1}{c}\frac{\partial f}{\partial t}, \quad \vec{A}' = \vec{A} + \nabla f, \tag{8}$$

where $f(\vec{x},t)$ is a function. **(2 points)**

(b) In a semiclassical model of a charged particle interacting with the electromagnetic field, we treat the particle as a quantum object, but still treat the potential ϕ and the vector potential $\vec{A}$ as classical (i.e., just being real and vector-valued functions of position $\vec{x}$ and time t). A standard choice of Hamiltonian is $H = \frac{1}{2m}\big(\vec{P} - \frac{e}{c}\vec{A}(\vec{X},t)\big)^2 + e\phi(\vec{X},t)$, and the time-dependent Schrödinger equation does in this case become

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = \frac{1}{2m}\big(-i\hbar\nabla - \frac{e}{c}\vec{A}(\vec{x},t)\big)^2\psi(\vec{x},t) + e\phi(\vec{x},t)\psi(\vec{x},t). \tag{9}$$

Suppose that $\psi(\vec{x},t)$ is a solution to (9). *Show that $\psi'(\vec{x},t) = \mathrm{e}^{i\frac{e}{\hbar c}f(\vec{x},t)}\psi(\vec{x},t)$ is a solution to*

$$i\hbar\frac{\partial}{\partial t}\psi'(\vec{x},t) = \frac{1}{2m}\big(-i\hbar\nabla - \frac{e}{c}\vec{A}'(\vec{x},t)\big)^2\psi'(\vec{x},t) + e\phi'(\vec{x},t)\psi'(\vec{x},t), \tag{10}$$

where ϕ' and $\vec{A}'$ are as in (8).

Hint: Keep in mind that $\big(-i\hbar\nabla - \frac{e}{c}\vec{A}(\vec{x},t)\big)^2\psi(\vec{x},t) = \big(-i\hbar\nabla - \frac{e}{c}\vec{A}(\vec{x},t)\big)\cdot\big(-i\hbar\nabla - \frac{e}{c}\vec{A}(\vec{x},t)\big)\psi(\vec{x},t)$, and that one should be careful to let the ∇ operate on everything they should operate on. **(6 points)**

Remark: This illustrates that when we gauge-transform the fields, then the wave-function of the particle also has to be transformed.

Gold-star exercise

This is a gold-star exercise, which means that you get no points what so ever for it, but you get a gold-star!

Back again to the Kitaev chain. Suppose that we instead have a chain with periodic boundary conditions, i.e., the total Hamiltonian would be

$$H' = H + c_1^\dagger c_N + c_N^\dagger c_1 + c_N^\dagger c_1^\dagger + c_1 c_N, \tag{11}$$

where H is as in (1). *What would the ground state degeneracy be in this case?*